Boston Physics Circle: Math + Dimensional Analysis notes

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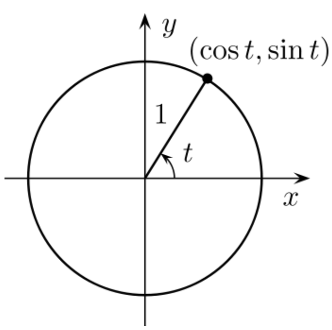
1 Introduction

This week we will go over some mathematical concepts and tools that will prove very useful further into your preparation for the F=ma exam, as well as your future studies in Physics and related areas.

2 Some Useful Math

2.1 Trigonometry

It is useful sometimes to switch to a different units of measuring angles from the one that we are used to: degrees. As you know the full angle is defined to be 360 degrees, the right angle 90 degrees et cetera. A different way of measuring angles is doing it in **Radians**.



Defitinion: The set of all points in the (x, y) plane that satisfy $x^2 + y^2 = 1$

Definition: An angle is θ radians if the arc that the angle spans in the unit circle is of length θ

Definition:
$\sin(t)$ is defined as the y coordinate of the point in the unit circle that makes and angle t with the x axis.
$\cos(t)$ is defined as the x coordinate of the point in the unit circle that makes and angle t with the x axis.
$\tan(t)$ is defined as simply,

$$\tan(t) \equiv \frac{\sin(t)}{\cos(t)}$$

You might also encounter the following definitions that are less common that the previous three but are still used for convenience.

$$\cot(t) \equiv \frac{1}{\tan(t)}, \quad \sec(t) \equiv \frac{1}{\cos(t)}, \quad \csc(t) \equiv \frac{1}{sin(t)}$$

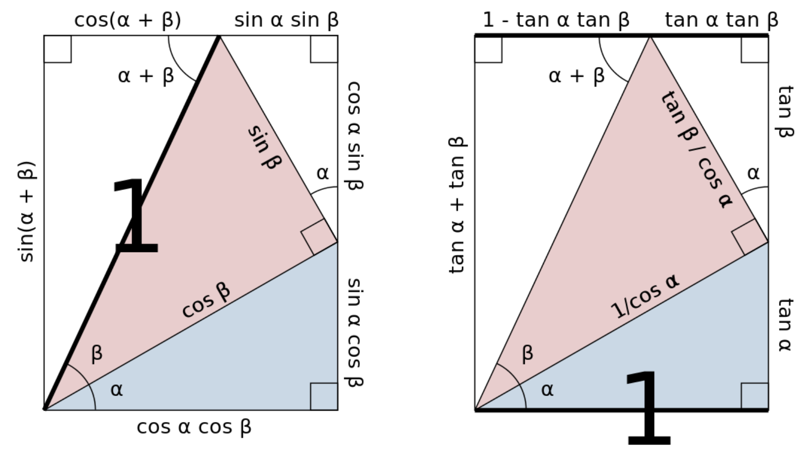
Discuss: You might have been introduced to *sin* and *cos* with the right triangle instead of the unit circle. The right triangle is a bad (limited) way of defining them. Why do you think that is the case?

A trivial identity that comes from the definition of the unit circle and the trigonometric functions is,

$$\sin(t)^2 + \cos(t)^2 = 1 \quad \text{for all t.}$$

Exercise: Try to express $tan(t)^2$ in terms of only $sin(t)^2$, then only $cos(t)^2$

Let us now derive the identities for the sum of two angles. If we draw useful rectangles with triangles in them.



(a) $\sin(\alpha + \beta)$ (b) $\tan(\alpha + \beta)$

These two can be seen from the geometric proofs above

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

Exercise: Derive the following by using what we have learned so far

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\sin(2u) = 2 \sin u \cos u$$

$$\cos(2u) = \cos(u)^2 - \sin(u)^2$$

$$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 - \cos u}{2}}$$

$$\cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan(\frac{\eta + \theta}{2}) = \frac{\sin \eta + \sin \theta}{\cos \eta + \cos \theta}$$

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\cot(2\theta) = \frac{\cot^2 \theta - 1}{2 \cot \theta}$$

$$2 \cos \theta \cos \varphi = \cos(\theta - \varphi) + \cos(\theta + \varphi)$$

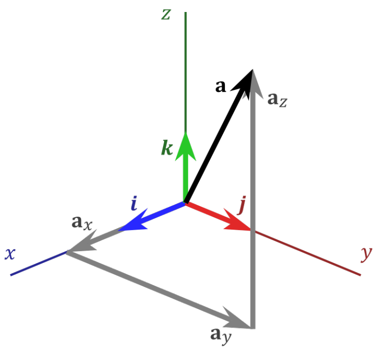
$$2 \sin \theta \sin \varphi = \cos(\theta - \varphi) - \cos(\theta + \varphi)$$

$$2 \sin \theta \cos \varphi = \sin(\theta + \varphi) + \sin(\theta - \varphi)$$

$$2 \cos \theta \sin \varphi = \sin(\theta + \varphi) - \sin(\theta - \varphi)$$

$$tan\theta \tan \varphi = \frac{\cos(\theta - \varphi) - \cos(\theta + \varphi)}{\cos(\theta - \varphi) + \cos(\theta + \varphi)}$$

2.2 Vectors and Vector Algebra



There are quantities in physics that require information about both magnitude and direction in order to be conveyed fully. A typical example is given as the velocity of a moving object. It is not enough to know that the object is traveling with 50 Mph. We also need to know which way that object is traveling to get the full picture. Such quantities are mathematically described with **vectors**. Quantities that only require a single number to carry all the information are called **scalars**.

Discuss Try to think of examples of scalar and vector quantities in physics.

The three basis vectors of 3 dimensional Euclidean space that are most commonly used are,

$$\mathbf{e}_1 = (1,0,0),\ \mathbf{e}_2 = (0,1,0),\ \mathbf{e}_3 = (0,0,1)$$

We can express any vector in 3D space as a linear combination of these vectors. Two vectors **a** and **b** can be written as,

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

$$\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3.$$

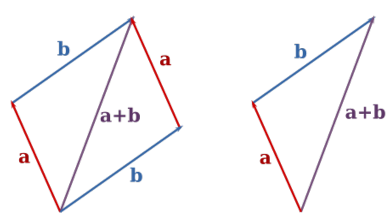
where a_i and b_i are real numbers that are called **components** of a vector. Two vectors are equal if all of their components are equal.

$$\mathbf{a} = \mathbf{b} \Longleftrightarrow a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3.$$

Sum of two vectors is a vector whose components are the sum of the two vectors' components.

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{e}_1 + (a_2 + b_2)\mathbf{e}_2 + (a_3 + b_3)\mathbf{e}_3.$$

This addition can also be represented visually by drawing the vectors as arrows and attaching the tip of one to the bottom of the other.



(c) adding vectors

Multiplying a vector with a scalar results in a vector whose components are the components of the previous vector multiplied by the scalar.

$$r\mathbf{a} = (ra_1)\mathbf{e}_1 + (ra_2)\mathbf{e}_2 + (ra_3)\mathbf{e}_3.$$

We can note here that this also automatically gives us the definition of subtraction,

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b} = (a_1 - b_1)\mathbf{e}_1 + (a_2 - b_2)\mathbf{e}_2 + (a_3 - b_3)\mathbf{e}_3$$

We can also define a form of multiplication for vectors which we will call the **dot product**. Unlike addition and subtraction, this dot product produces a scalar instead of a vector.

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Now the **magnitude**, or **length**, or **norm** of a vector can be defined simply as,

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Exercise: Show that the dot product of two vectors can also be written as,

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

Where θ is the angle between two vectors. **Hint:** Without the loss of generality, you can choose one of the vectors to lie in the x axis and the other to lie on the x-y plane. So the z components of both are 0 and the y component of the first vector is 0.

Discuss: What might be the geometric meaning of the dot product. Can you think of something concrete that it calculates?

Exercise: Show that the dot product has the following properties:

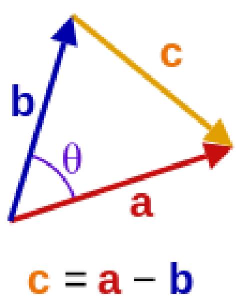
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$\mathbf{a} \cdot (r\mathbf{b} + \mathbf{c}) = r(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$$

$$(c_1\mathbf{a}) \cdot (c_2\mathbf{b}) = c_1 c_2 (\mathbf{a} \cdot \mathbf{b})$$

Using the dot product we can prove a very useful geometric identity called **law of cosine**, which can be thought of as a generalized Pythagoras' Theorem.



(d) cosine law

$$\begin{aligned}
\mathbf{c} \cdot \mathbf{c} &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\
&= a^2 - \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} + b^2 \\
&= a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2 \\
c^2 &= a^2 + b^2 - 2ab\cos\theta
\end{aligned}$$

We can also define a vector multiplication that gives us another vector instead of a scalar. The **cross product** of two vectors is defined as,

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\mathbf{e}_3$$

It can also be written as,

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \, \mathbf{n}$$

Where θ is the angle between two vectors and $\mathbf{n}$ is a unit vector that is perpendicular to both of the vectors. There are two such unit vectors and they face opposite directions. By convection we choose the one that satisfies the **right hand rule**.

Exercise: Show that both of the defitions above are equivalent, like you have shown earlier in the case of the dot product. You should use the same hint.

2.3 Useful Approximations

In physics, sometimes we cannot solve problems fully. However, we can make very useful approximations that give us answers that are valid for all practical purposes. You can think of this as measuring the length of a table. You only need to know it up to .1 inches. Any more accuracy is redundant.

$$(1+x)^n \approx (1+nx) \text{ for } |x| < 1 \text{ and } |nx| \ll 1$$

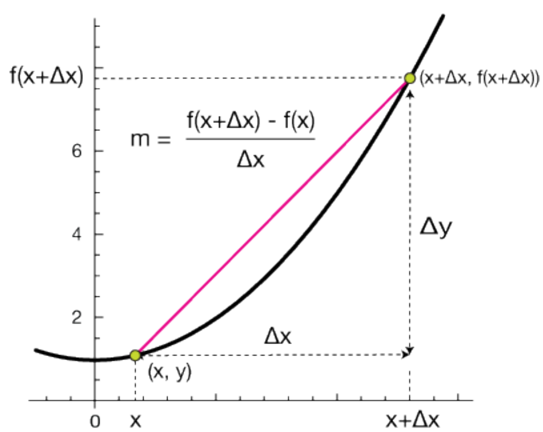
$$\sin x \approx \tan x \approx x \text{ for } x \ll 1$$

,

$$\cos x \approx 1 - \frac{x^2}{2} \text{ for } x \ll 1$$

Discuss: Try to see how these approximations come about. Some can be understood geometrically.

2.4 Concept of the Derivative



(e) derivative

The derivative of a function f at point x is defined as,

$$\frac{df}{dx}(x) \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example Let $f(x) = x^2$. Then the derivative of f at x is,

$$\frac{df}{dx}(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + \Delta x^2 + 2x\Delta x - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x^2}{\Delta x} + \lim_{\Delta x \to 0} \frac{2x\Delta x}{\Delta x}$$
$$\frac{df}{dx}(x) = 0 + 2x = 2x$$

The mathematical concept of the derivative is very useful to physics because it enables us to define the velocity or the acceleration of an object at a moment in time. Without derivatives we would only have average velocities and average accelerations.

Let us have an object that moves only in a single dimension x. Let us denote it's position as a function of time as $x(t) =$ it's position at time t. The velocity of the object at time t then is simply the derivative of $x(t)$ with respect to t, denoted as,

$$v(t) \equiv \frac{dx}{dt}(t)$$

It's acceleration at time t is simply the derivative of $v(t)$ with respect to t, denoted as,

$$a(t) \equiv \frac{dv}{dt}(t) = \frac{d\frac{dx}{dt}}{dt}(t) = \frac{d^2x}{dt^2}(t)$$

Exercise: Now that you have learned how manipulate vectors, try to extend these single dimensional concepts of position, velocity, and acceleration to 3D space.

3 Dimensional Analysis

In physics, when we want to find out how quantity of interest is affected by other parameters. For example we can ask when someone is skydiving, how is one's terminal velocity dependent on one's mass, keeping everything else unchanged. Dimensional analysis stems from the idea that any meaningful equation will have the same units on both sides.

$$A = B \Longrightarrow [A] = [B]$$

where $[A]$ means the dimension(unit) of A. We can look at the famous equation $F = ma$ as an example and check,

$$[F] = \text{Newtons} = kg\frac{m}{s^2}, \quad [ma] = [m][a] = kg\frac{m}{s^2}$$

SI standard for dimensions: length (L), mass (M), time (T), electric current (I), absolute temperature (Θ), amount of substance (N) and luminous intensity (J).Mathematically, the dimension of the quantity Q is given by,

$$[Q] = \mathsf{L}^a \mathsf{M}^b \mathsf{T}^c \mathsf{I}^d \Theta^e \mathsf{N}^f \mathsf{J}^g$$

quantity For every quantity there is a combination for the exponents a, b, c, d, e, f, g.

Now let us see what we can do with this tool. Consider the case of water waves. We know that their speed is only dependent on depth H, wavelength λ, and gravitational acceleration g. The only way to combine these quantities such that they give the unit of velocity is,

$$v = \sqrt{gH} f\left(\frac{\lambda}{H}\right)$$

where f is a dimensionless function. In deep water we expect the velocity to be independent of depth. The only f that satisfies that is,

$$f\left(\frac{\lambda}{H}\right) = K\sqrt{\frac{\lambda}{H}}$$

Where K is a dimensionless constant. So we get,

$$v_{deep} = K\sqrt{g\lambda}$$

So in deep waters waves with longer wavelengths travel much faster.

In shallow water we expect the velocity to be independent of wavelength so $f = C = const$, which gives,

$$v_{shallow} = C\sqrt{gH}$$

Discuss: Ass the waves approach the shore, their velocity decreases because the water gets shallower. Using this fact, can you explain why we never see waves crash the shore with very large angles, e.g. 80 degrees?

Now consider a spring with a spring constant k attached to the ceiling on one end, and to a ball of mass M on the other end. How is the period T of the vertical oscillation of this ball dependent on M and k? With a dimensionless constant C, and exponents p and q that we want to find, we can write,

$$T = C(M)^p(k)^q$$

$$[T] = [M]^p[k]^q$$

The equation for the spring constant is,

$$k = F/\Delta x \text{ which implies } [k] = Newton/m = kg/s^2$$

$$s = kg^p(kg/s^2)^q = kg^{p+q}s^{-2q}$$

$$0 = p + q \text{ and } 1 = -2q \text{ gives } q = -1/2 \text{ and } p = 1/2$$

$$T = C\sqrt{M/k}$$

Exercise: The terminal velocity of a skydiver is dependent on the diver's mass M and surface area A, the air density ρ, and the gravitational acceleration g. Using dimensional analysis, find out the formula (up to a dimensionless constant) for the free fall velocity.