Harvard Physics Circle
Circular Motion

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1 Core Principles

For an object moving in a circle of radius r with speed v, its angular velocity is $\omega = v/r$. Its centripetal acceleration is $a_C = r\omega^2 = v^2/r$.

For a rigid body, the torque τ about the center of mass is related to the angular acceleration α by $\tau = I\alpha$, where I is the moment of inertia.

Schematically, the torque is $\tau = \sum_i \mathbf{r}_i \times \mathbf{F}_i$ and the moment of inertia is $I = \sum_i m_i r_i^2$.

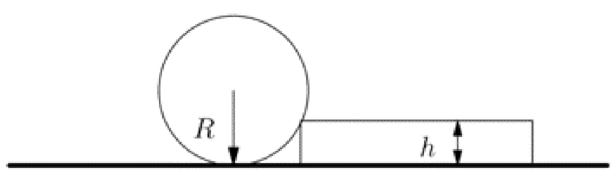
The rotational kinetic energy of a rigid body with angular velocity ω_{CM} about its center of mass and moment of inertia I_{CM} is $K_{\mathsf{rot}} = I_{\mathsf{CM}}\omega_{\mathsf{CM}}^2/2$.

2 $F = ma$ Exam Questions

Remarks: For the rest of this document, [...]* denotes priority examples. Other problems are recommended examples – we will go over them if we have time.

[2019A, Q5]*

5. A cylinder has a radius R and weight G. You try to roll it over a step of height $h < R$. The minimum force needed to roll the cylinder over is:



(A) $\dfrac{\sqrt{2Rh - h^2}}{R - h}G$

(B) $\dfrac{\sqrt{2Rh - h^2}}{2R - h}G$

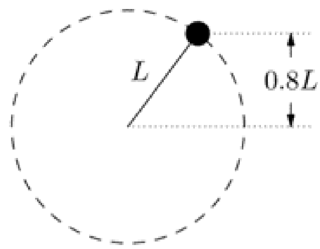
(C) $\dfrac{\sqrt{2Rh - h^2}}{2R}G$

(D) $\dfrac{\sqrt{2Rh - h^2}}{R}G$

(E) $\dfrac{\sqrt{Rh - h^2}}{2R}G$

[2019A, Q7]*

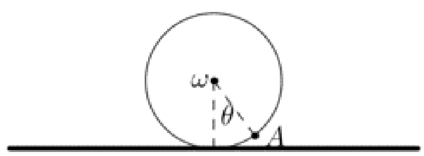
7. An object of mass m is attached to the end of a massless rod of length L. The other end of the rod is attached to a frictionless pivot. The object is raised so that its height is $0.8L$ above the pivot, as shown in the figure. After the object is released from rest, what is the tension in the rod when it is horizontal?



(A) $0.6\ mg$
(B) $1.6\ mg$
(C) $2.6\ mg$
(D) $3.6\ mg$
(E) $5.36\ mg$

[2019A, Q9]*

9. A wheel of radius R is rolling without slipping with angular velocity ω.



For point A on the wheel at an angle θ with respect to the vertical, shown in the figure, what is the magnitude of its velocity with respect to the ground?

(A) ωR
(B) $\omega R \sin(|\theta|/2)$
(C) $\sqrt{2}\omega R \sin(|\theta|/2)$
(D) $2\omega R \sin(|\theta|)$
(E) $2\omega R \sin(|\theta|/2)$

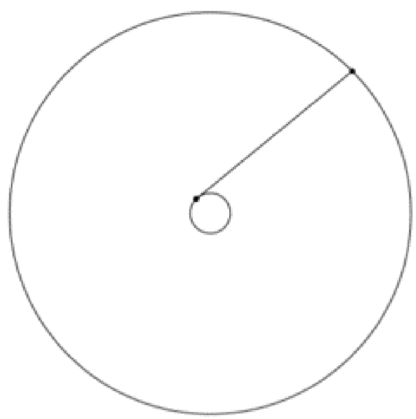
[2019A, Q10]*

10. A flat uniform disk of radius $2R$ has a hole of radius R removed from the center. The resulting annulus is then cut in half along the diameter. The remaining shape has mass M. What is the moment of inertia of this shape, about the axis of rotational symmetry of the original disk?

    (A) $\frac{45}{32}MR^2$
    (B) $\frac{7}{6}MR^2$
    (C) $\frac{8}{5}MR^2$
    (D) $\frac{5}{2}MR^2$
    (E) $\frac{15}{8}MR^2$

[2019B, Q15]

15. A mass of $M = 100$ g is attached to the end of a string of length $R = 2$ m. A person swings the mass overhead such that their hand traverses a circle of radius $r = 3$ cm at angular velocity $\omega = 10$ rad/s, ahead of the mass M by an angle of $\pi/2$. Estimate the force of air resistance on the object.

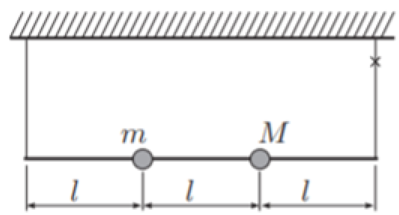


    (A) 0.02 N
    (B) 0.03 N
    (C) 0.2 N
    (D) 0.3 N
    (E) 2 N

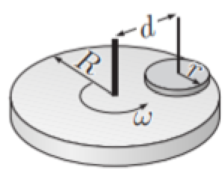
3 Longer Problems

Remarks: [Kalda] refers to https://www.ioc.ee/~kalda/ipho/meh_ENG.pdf. [IPhO] refers to the International Physics Olympiad; past problems can be found in many places online. Problems labeled [N/A] have no official source.

[Kalda, P30]*
A light rod with length $3l$ is attached to the ceiling by two strings of equal length. Two balls with masses m and M are fixed to the rod, the distance between them and their distances from the ends of the rod are all equal to l. Find the tension in the left string right after the right string has been cut.



[Kalda, P18]*
A horizontal platform rotates around a vertical axis at angular velocity ω. A disk with radius r can freely rotate and move vertically along a slippery vertical axle situated at a distance $d > r$ from the platform's axis. The disk is pressed against the rotating platform due to gravity, and the coefficient of friction between the surfaces is μ. Find the angular velocity acquired by the disk. Assume that pressure is distributed evenly over the entire base of the disk.



[IPhO, 1998 Theory P1a]*
Consider a long, solid, rigid and regular hexagonal prism, like a common type of pencil. The prism has mass M and uniform density. The length of each side of the cross-sectional hexagon is a.

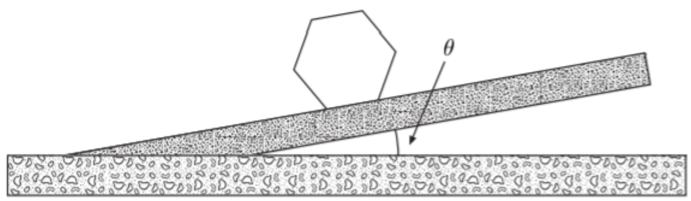
It is given that the moment of inertia I of the hexagonal prism about its central axis is

$$I = \frac{5}{12} M a^2. \tag{1}$$

Part 1: Show that the moment of inertia about an edge of the prism is

$$I' = \frac{7}{12} M a^2. \tag{2}$$

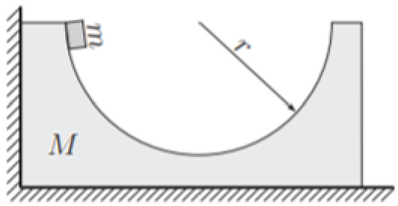
The prism is initially at rest with its axis horizontal on an inclined plane that makes a small angle θ with the horizontal.



The prism is now displaced from rest and starts unevenly rolling down the plane. Assume that friction prevents any sliding and that the prism does not lose contact with the plane. The angular velocity just before a given edge hits the plane is ω_i, while the angular velocity immediately after the impact is ω_f.

Part 2: Show that we may write $\omega_f = s\omega_i$ and find the value of the coefficient s.

[Kalda, P29]
A block with mass M lies on a horizontal slippery surface and also touches a vertical wall. In the upper surface of the block, there is a cavity with the shape of a half-cylinder with radius r. A small pellet with mass m is released at the upper edge of the cavity, on the side closer to the wall. What is the maximum velocity of the block during its subsequent motion? Neglect frictional effects.



[N/A]
The equation $\tau = I\alpha$ conceals a complicated property about the moment of inertia: the torque τ and angular acceleration $\alpha = \dot{\omega}$ need not always be in the same direction. This means that the moment of inertia I is really a 3×3 matrix.

Show that, for a point mass m at position (x, y, z) relative to the origin O, the angular momentum and torque are related by

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = m \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}.$$