Harvard Physics Circle

Coulomb's Law, Gauss's Law, Electric Potential

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§1 Coulomb's Law

The typical example when used to describe an electrostatic setting, is to consider two point changes, of charge q_1 and q_2 respectively. Coulomb's law in electrostatics gives the interaction force between these two charges, as a function of the distance, r, between them. The force is repulsive if the two charges are both negative or positive, and it is attractive if they have different signs. It is equivalent to the attraction forces between two point masses in mechanics and has the formula:

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon} \frac{\vec{r_1} - \vec{r_2}}{|\vec{r_1} - \vec{r_2}|^3} \qquad |\vec{F}| = \frac{q_1 q_2}{4\pi\epsilon} \frac{1}{r^2}$$

In the expression above, $k_e = \frac{1}{4\pi\epsilon} \approx 9 \times 10^9 N \cdot m^2 \cdot C^{-2}$ is called the Coulomb constant, where ϵ is the electric constant (or permittivity) of the medium the charges are in. For vacuum, it is denoted with $\epsilon_0 = 8.854 \times 10^{-12}$ F/m (Farads per meter). For other material, $\epsilon = \epsilon_0 \cdot \epsilon_r$, where ϵ_r is the relative permittivity and is usually greater than 1.

§2 Gauss's Law

To understand the thinking behind the formula we are about to give, let us first define the electrical flux,

$$\Phi_E = \int_S \vec{E} \cdot \hat{n} \cdot dS$$

,where S is the surface through which the electric field passes, and $\hat{n}$ is the unit vector representing the normal of this surface. We can write Gauss's law in two ways, which represent the same concept: the net electric flux through any closed surface is proportional to the charge contained by the volume bounded by that surface. The <u>Integral Form</u> is as follows:

$$\Phi_E = \int_S \vec{E} \cdot \hat{n} \cdot dS = \int_V \nabla \cdot \vec{E} dV = \frac{Q}{\epsilon_0}$$

If we are not dealing with a point charge, but with a continuously distributed charge, i.e. $Q = \int_V \rho dV$, then we can write it in the <u>Differential Form</u> as:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Note that in both formulas, $\nabla \cdot \vec{E}$ refers to the dot product of the divergence with the electric field (such that we end up with a scalar, unlike the gradient or vector product).

§3 Electric Potential

The difference between the Electric Potential and the gravitational potential is similar to the difference between Coulomb's law for the electrostatic force and the attraction force in mechanics. The formula for the electric potential given by a charge Q at a distance r from it is:

$$V = \frac{1}{4\pi\epsilon} \frac{Q}{r}$$

Alternatively, it can also be defined as the integral of the electric field E on a contour line C:

$$V = \int_C E \cdot dl$$

§4 Problems

§4.1 Motion of charged particles - Russian Physics Olympiad 2010

Four point masses m freely float in space, being initially at the corners of a square inscribed by a circle of radius R_0. Two charges have electrical charge $+q$, and the other two have charge $-q$. Initially, the masses were given the same clockwise velocities tangent to the circle. It is known that the minimum distance between a mass and the circle center O during the motion equals $R_1 < R_0$. Suppose the systems remains symmetrical with respect to the axis O during the time of observation. Gravity is negligible. Determine the mass trajectories and the period of their motion.
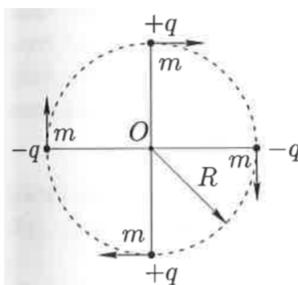

§4.2 Charge above square plate - 200 More Puzzling Physics Problems, Problem 136

A square of side d, made from a thin insulating plate, is uniformly charged and carries a total charge of Q. A point charge q is placed on the symmetrical normal axis of the square at a distance $d/2$ from the plate. How large is the force acting on the point charge?
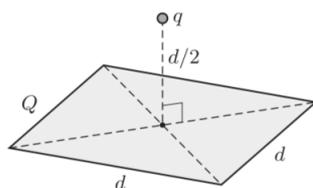
§4.3 Cubic oscillations - IPhO 2020, Problem 1A

A narrow straight channel passes through the center of a fixed cube with a side a. The cube is uniformly charged, the charge density is ρ. The distance from the cube center to the point of intersection of the channel and a face is L. In the channel there is a particle of a mass m and a charge q Find the period of small oscillations of the particle near the center. The gravitational interaction of the particle and the cube can be neglected. The cube and the particle are oppositely charged.
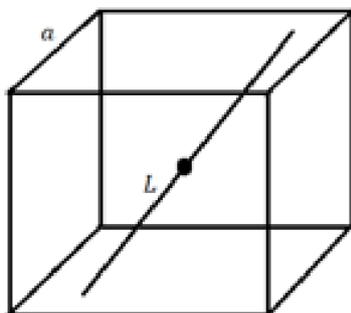
§4.4 Electric field inside a charged sphere with impurity - IPhO 2013, Problem 2.2

For the rest of the problem assume that the relative dielectric permittivity of all materials is 1. Inside a charged sphere of homogeneous charge density ρ and radius R is created a small spherical charge-neutral region of radius R_1 by adding the opposite charge density $-\rho$, with its center displaced by from the center of the R-sphere by $\vec{x}_d = x_d \vec{e}_x$, see Fig. (b). Show that the electric field inside the charge-neutral region is homogenous of the form $\vec{E} = A(\rho/\epsilon_0)\vec{x}_d$, and determine the pre-factor A.
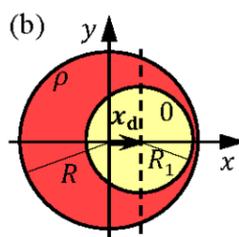
§5 Practice Problems

## §5.1 Two insulating rods- 200 More Puzzling Physics Problems, Problem 134

Two very long, thin, insulating rods, each carrying uniform linear charge density λ, lie in perpendicular directions (in three-dimensional space), a distance d from each other (see figure). What is the magnitude of the force of repulsion between them?
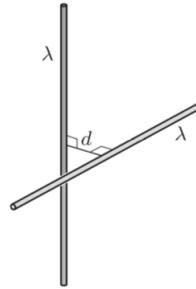
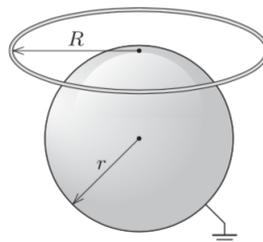## §5.2 Charged semi-spheres - 200 More Puzzling Physics Problems, Problem 140

Two insulating hemispherical shells (e.g. the two halves of a ping- pong ball) are placed very close to each other with their centres almost coincident, as shown in figure a). Both of them are uniformly charged, one with total electric charge Q, and the other with total charge q.

a) What is the magnitude of the net electric force exerted by the two bodies on each other?

b) (PRACTICE) Does the result change if the radius of one of the hemispherical shells is only one-half that of the other, as shown in figure b)?
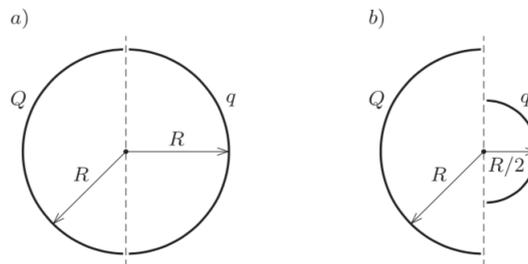

§5.3 Ring above grounded sphere - 200 More Puzzling Physics Problems, Problem 151

A thin metal ring of radius R is charged in such a way that the electric potential at its centre is V_0. The ring is now placed horizontally above a grounded solid metal sphere of radius r, so that the centre of the ring and the top of the sphere coincide. Find the total charge induced on the sphere.

4

§5.4 Pyramid molecule - 200 More Puzzling Physics Problems, Problem 152

Four identical non-touching metal spheres are positioned at the vertices of a regular tetrahedron, as shown in the figure. A charge of 20 nC given to one of the spheres, A, raises it to a potential V. Sphere A can also be raised to potential V if it and one of the other spheres are each charged with 15 nC. What must be the size of equal charges given to A and two other spheres for the potential of A to again be raised to V? And if all four spheres are used?
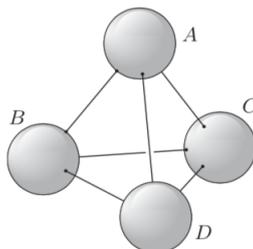
§5.5 Charged cube - 200 More Puzzling Physics Problems, Problem 137

(Nice symmetry) A cube is made from six thin insulating square faces, each square having side d and carrying a uniformly distributed charge of Q. How large is the electrostatic force acting on each face?

§6 Recommendations from International Competitions

From IPhO:

1. Problem 2, 2013 - Plasmonic Steam Generator.
   Problem: https://www.ioc.ee/~kree/students/iphoTable/files/ipho/2013_Denmark_p2.pdf,
   Solution: https://www.ioc.ee/~kree/students/iphoTable/files/ipho/2013_Denmark_p2Sol.pdf

2. Problem 1, 2004 - 'Ping- Pong' Resistor.
   Problem: https://www.ioc.ee/~kree/students/iphoTable/files/ipho/2004_Korea_p1.pdf,
   Solution: https://www.ioc.ee/~kree/students/iphoTable/files/ipho/2004_Korea_p1Sol.pdf

From APhO:

1. Problem 1, 2011 - The Shockley-James Paradox. (requires knowledge of magnetism)
   Problem: http://staff.ustc.edu.cn/~bjye/ye/APhO/2011Q1.pdf,
   Solution: http://staff.ustc.edu.cn/~bjye/ye/APhO/2011S1.pdf

§7 References

1. Wikipedia, Divergence - formulas for all systems of coordinates
   https://en.wikipedia.org/wiki/Divergence

2. Péter Gnädig, Gyula Honyek, K. F Riley. 200 More Puzzling Physics Problems.

3. Jaan Kalda. IPhO Problems. https://physprob.com/