Faraday's Law, Inductors, and RLC Circuits, Harvard Physics Circle

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1 Review

This review is quite brief. If you are unfamiliar with the topic, please read chapter 7 of [PM].

1.1 Faraday's Law

Suppose we have a configuration of electric and magnetic fields (potentially changing over time), and a loop C in space (potentially changing over time). The EMF about the loop is defined to be

$$\mathcal{E} = \oint_C \tfrac{\vec{F}}{q} \cdot \vec{d\ell},$$

where $\frac{\vec{F}}{q}$ is the force per unit charge due to the fields at a given point. Faraday's law states that

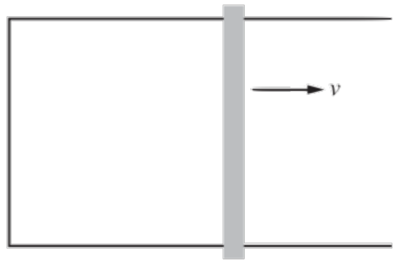
$$\mathcal{E} = -\frac{d\Phi}{dt},$$

where Φ is the flux of the magnetic field through C. The negative sign indicates that the direction of the emf would produce a current in the loop that opposes the change in flux.

Side note: Another more mathematical way to keep track of the signs is to make sure that the direction of line integration along C and the unit normal vector direction when computing the flux through a surface bounded by C are compatible according to the right hand rule (i.e. if we curl our right hand along the line integration direction, our thumb should point in the direction of the normal vector for computing the flux). With this convention, the negative sign in Faraday's law will take care of everything directly.

1.1.1 Fixed Fields, Moving Loop

We suppose that the fields are stationary in time and the loop is moving. It turns out that Faraday's law in this case can be derived directly from the Lorentz force laws. The following example illustrates this point.



Consider two conducting rails as shown above (with minimal resistance), and a sliding bar on them. If the bar is maintained to move at speed v to the right, and the bar has resistance R, what is the current through the bar?

The first approach is to cite Faraday's law, which gives us that

$$|\mathcal{E}| = BvL,$$

where L is the length of the bar, so

$$I = |\mathcal{E}|/R = BvL/R.$$

We can also see this through the Lorentz force law. Indeed, the component of the force on an electron in the rod parallel to the rod is

$$F = evB,$$

so the so there is an effective potential of vBL across the rod, giving the same result as above.

1.1.2 Time Varying Fields, Fixed Loops

In this case, the EMF $\mathcal{E}$ is generated entirely by a *nonconservative* electric field. In particular, Faraday's law states that if C is a fixed loop bounding a surface S, then

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{A}.$$

Unlike the previous case of Faraday's law, this is new physics.

1.2 Inductance

The *inductance* is a property of a conducting loop, and it only depends on the geometry of the loop. The inductance is simply defined to be the coefficient L such that

$$\Phi = LI,$$

where I is a current flowing through the loop, and Φ is the flux of the induced magnetic field. Faraday's law states that there is an effective voltage drop of $-L\dot{I}$ across an inductor. The energy stored in an inductor is

$$U = \frac{1}{2}LI^2.$$

The quantity L is also referred to as the self-inductance. If we have two conducting loops C_1 and C_2, then there is also a mutual inductance M, which is defined so that

$$\Phi_1 = L_1 I_1 + M I_2$$

and

$$\Phi_2 = M I_1 + L_2 I_2.$$

The fact that there exists such an M is nontrivial (in particular the fact that the two equations have the same coefficient M), but can be shown to be true.

1.3 RLC Circuits

We can also look at an inductor as a circuit element, similar to a capacitor or a resistor. The most common example is when we have resistors, capacitors, and inductors in series with a battery.

1.3.1 RC Circuits

Suppose we have a capacitor C and a resistor R in series with a battery of voltage V. If Q is the charge on the positive plate of the capacitor, then we have

$$V = \tfrac{Q}{C} + RI,$$

and $I = \dot{Q}$. Thus,

$$\tfrac{1}{C}I + R\dot{I} = 0,$$

so

$$I = I_0 e^{-t/\tau}$$

where $\tau = \frac{1}{RC}$ is the *RC time constant.*

1.3.2 RL Circuits

Suppose we have an indcutor L and a resistor R in series with a battery of voltage V. We have

$$V = L\dot{I} + RI,$$

So,

$$I = I_1 + I_0 e^{-t/\tau}$$

where $\tau = \frac{L}{R}$ is the *LR time constant* (note that there is a relation between I_1 and I_0 found when plugging back in).

1.3.3 RLC Circuits

Suppose now we have all three elements R, L, C in series connected to a battery of voltage V. We have

$$V = \frac{Q}{C} + RI + L\dot{I},$$

so

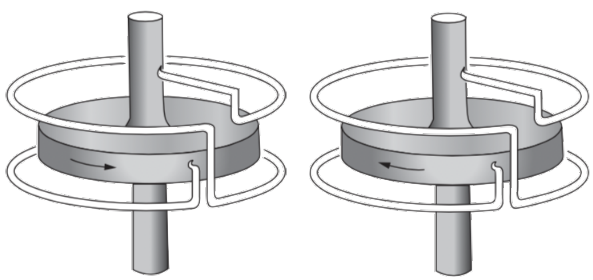
$$\tfrac{1}{C}I + R\dot{I} + L\ddot{I} = 0.$$

This is damped simple harmonic motion. If the resistance is small, we can approximate this as regular simple harmonic motion, which is commonly known as the LC case.

2 Problems

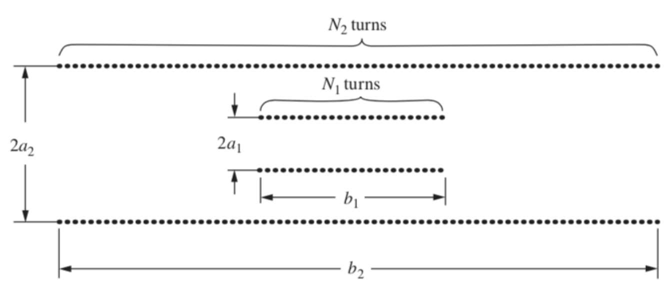
Problem 1 ([PM] 7.47)**.** A self-excited dynamo works as follows: it is a conductor that is driven through a magnetic field, generating an EMF, which causes current to flow in the conductor. The tricky part is that the origin of the magnetic field is the current itself, which explains why the dynamo is "self-excited".

Which of the following two pictures could be a dynamo?



Problem 2 (USAPhO 2009 A1)**.** See the end of this document for problem statement.

Problem 3 ([PM] 7.9)**.** Find the mutual inductance of the two solenoids shown below, where $a_1 \ll b_1$, $a_2 \ll b_2$, and $b_1 \ll b_2$.



Problem 4 ([MPPP] 181)**.** Three circular loops with radii R, $2R$, and $4R$ are placed concentrically. The inner and outer ones are not completed, but instead have two nodes. A

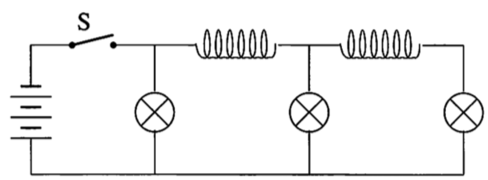
time varying current flows through the middle loop, and a certain point in time, the voltage across the nodes of the inner loops is V_0. What is the voltage across the nodes of the outer loop?

Problem 5 (Classical)**.** Prove the stationary-field case of Faraday's law with just the Biot-Savart law (requires some vector calculus).

Problem 6 (Classical)**.** Prove the symmetry of mutual inductance (as explained in the section on mutual inductance).

Problem 7 (Classical)**.** Given two inductors L_1 and L_2 with mutual inductance M, prove that $M^2 \leq L_1 L_2$ (hint: the amount of work required to create a setup with currents I_1 and I_2 must be positive, otherwise we get free electricity).

Problem 8 ([PPP] 171)**.** A circuit contains three identical lamps (i.e. resistors) and two identical inductors, as shown below.



The switch S is closed for a long time, and then opened. Right after the opening, what are the relative brightnesses of the lamps?

3 More Practice

- [PPP] 177
- APhO 2009/2 (a more in depth analysis of a dynamo)
- IPhO 2014/1C

References

[PPP] Gnädig, Peter and Honyek Gyula and Riley, Ken. *200 Puzzling Physics Problems With Hints and Solutions.*

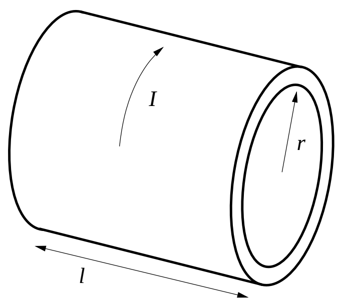
[MPPP] Gnädig, Peter and Honyek Gyula and Riley, Ken. *200 More Puzzling Physics Problems With Hints and Solutions.*

[PM] Purcell, Edward M. and Morin, David J. *Electricity and Magnetism.*

Part A

Question A1

A hollow cylinder has length l, radius r, and thickness d, where $l \gg r \gg d$, and is made of a material with resistivity ρ. A time-varying current I flows through the cylinder in the tangential direction. Assume the current is always uniformly distributed along the length of the cylinder. The cylinder is fixed so that it cannot move; assume that there are no externally generated magnetic fields during the time considered for the problems below.



a. What is the magnetic field strength B inside the cylinder in terms of I, the dimensions of the cylinder, and fundamental constants?

b. Relate the emf $\mathcal{E}$ developed along the circumference of the cylinder to the rate of change of the current $\frac{dI}{dt}$, the dimensions of the cylinder, and fundamental constants.

c. Relate $\mathcal{E}$ to the current I, the resistivity ρ, and the dimensions of the cylinder.

d. The current at $t = 0$ is I_0. What is the current $I(t)$ for $t > 0$?

Question A2

A mixture of ^{32}P and ^{35}S (two beta emitters widely used in biochemical research) is placed next to a detector and allowed to decay, resulting in the data below. The detector has equal sensitivity to the beta particles emitted by each isotope, and both isotopes decay into stable daughters.

You should analyze the data graphically. Error estimates are not required.

Day	Activity	Day	Activity	Day	Activity
0	64557	40	12441	200	1121
5	51714	60	6385	250	673
10	41444	80	3855	300	467
20	27020	100	2734		
30	18003	150	1626		

a. Determine the half-life of each isotope. ^{35}S has a significantly longer half-life than ^{32}P.

b. Determine the ratio of the number of ^{32}P atoms to the number of ^{35}S atoms in the original sample.