Harvard Physics Circle
Gravitational Laws

Yuan Lee
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1 Core Principles

Newton's law of gravitation states that the gravitational force $\mathbf{F}$ between two objects of mass m and M separated by a distance r has magnitude

$$F = \frac{GmM}{r^2}.$$

$G = 6.67 \times 10^{-11} \mathrm{N\,m^2\,kg^{-2}}$ is the universal gravitational constant.

The gravitational potential energy associated with the two-body system is $U = -\frac{GmM}{r}$. Note that the potential energy decreases as r decreases, indicating that the gravitational force is attractive.

In vector form, $\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) = -F\mathbf{r}/r$, where $\mathbf{r}$ is the displacement vector between the two masses m and M. (By definition, $r = |\mathbf{r}|$.)

2 Conceptual Problems

Remarks: For the rest of this document, [...]* denotes priority examples. Other problems are recommended examples – we will go over them if we have time.

[**Field strength**]*
Given a system of reference objects, we can define a *gravitational field* that describes the motion of a test mass under the influence of gravity.

The *gravitational field strength* g is the force experienced per unit mass due to this gravitational field. By Newton's second law, the field strength is also the acceleration due to gravity.

Verify that the gravitational field strength is $g = -GM/r^2$.

Given that the Earth has mass $M_E = 5.97 \times 10^{24}\,\mathrm{kg}$ and radius $R_E = 6370\,\mathrm{km}$, find the gravitational field strength at the Earth's surface (due to the Earth alone). Compare your answer to the commonly-quoted acceleration due to gravity $9.81\,\mathrm{m\,s^{-2}}$.

[**Acceleration due to gravity**]
Let g be the acceleration due to gravity on the Earth surface. Express, to first order in h, the force on a test mass m a small height $h \ll R_E$ above the Earth's surface.

[**Potential**]*
The *gravitational potential* V at location X is the work done per unit mass to move a test object from infinity to X without a change in speed.

Show that the gravitational potential around a body of mass M is $V = -GM/r$, where X is a distance r from the center of the body.

[Escape velocity]*

The *escape velocity* is the minimum speed needed for an object to escape the gravitational field of a (large) body.

Show that the escape velocity from the surface of the Earth is $\sqrt{2GM_E/R_E}$.

[Orbits]

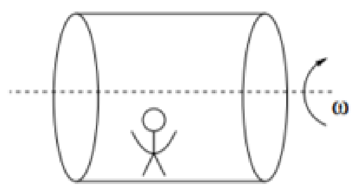
Show that the speed of an object orbiting the Earth with radius of orbit r is $\sqrt{GM_E/r}$.

3 $F = ma$ Exam Questions

Remarks: Most questions in the $F = ma$ exam that explicitly require the laws of gravity involve orbital motion, which is the subject of next week's physics circle. If you would like to read ahead, *Kepler's three laws of orbital motion* might be a helpful starting point.

[2018A, Q16]*

16. A cylindrical space station produces 'artificial gravity' by rotating with angular frequency ω. Consider working in the reference frame rotating with the space station. In this frame, an astronaut is initially at rest standing on the floor, facing in the direction that the space station is rotating. The astronaut jumps up vertically relative to the floor of the space station, with an initial speed less than that of the speed of the floor. Just after leaving the floor, the motion of the astronaut, relative to the space station floor,



(A) always has a component of acceleration directed toward the floor, and they land at the same point they jumped from.

(B) always has a component of acceleration directed toward the floor, and they land in front of the point they jumped from.

(C) always has a component of acceleration directed toward the floor, and they land behind the point they jumped from.

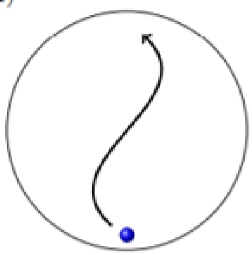
(D) has a component of acceleration directed away from the floor, and they land behind the point they jumped from.

(E) has a zero acceleration relative to the floor, and the astronaut never reaches the floor again.

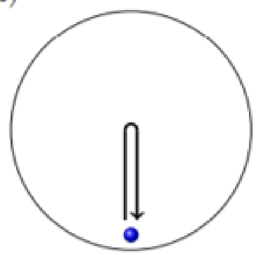
[2018B, Q12]*

12. A child in a circular, rotating space station tosses a ball in such a way so that once the station has rotated through one half rotation, the child catches the ball. From the child's point of view, which plot shows the trajectory of the ball? The child is at the bottom of the space station in the diagrams below, but only the initial location of the ball is shown.

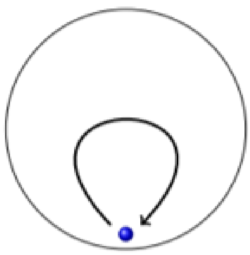
(A)



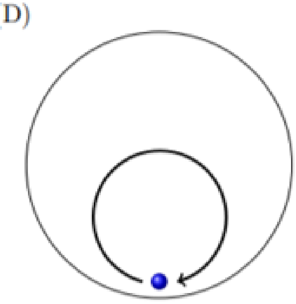
(B)



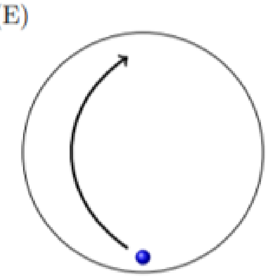
(C)



(D)



(E)



4 Longer Problems

[Schwarzchild radius]*
A *black hole* is an extremely dense astronomical object that has such a large gravitational field that (almost) nothing can escape. The *event horizon* of a black hole is its "point of no return"; we think information contained within the event horizon of a black hole is lost to observers outside the event horizon.

The event horizon is located at the *Schwarzschild radius* R_S of the black hole. One way to compute the Schwarzschild radius is to find the radius from which the escape velocity of any object is the speed of light. (The idea is that even light cannot escape from the black hole.)

Find the Schwarzschild radius of a (non-rotating, uncharged) black hole with mass M.

["Journey to the Center of the Earth"]*
Consider a tunnel dug along the diameter of the Earth, i.e. from one point on the Earth to the other diametrically opposite point, assuming the Earth is spherical. What is the motion of an object released from rest at one end of the tunnel?

[Morin, 5.12]*
Consider a cube of uniform mass density. Find the ratio of gravitational potential energy of a mass at a corner to that of the same mass at the center.

Hint: Use scaling laws.

[Morin, 5.13] – partly *
A hole of radius R is cut out from an infinite flat sheet with mass density σ per unit area. Let L be the line that is perpendicular to the sheet and that passes through the center of the hole.

*Part 1a:** When $R = 0$, what is the force on the mass m that is located on L at a distance x from the center of the hole?

Part 1b: For general R, find the force on the mass m located on L at a distance x from the center of the hole.

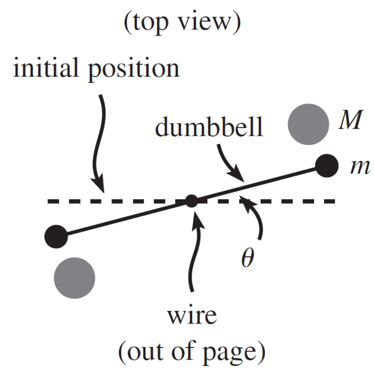
Part 2: If a particle is released from rest on L, very close to the center of the hole, show that it undergoes oscillatory motion and find the frequency of these oscillations.

*Part 3a:** Suppose a particle is released from rest on L at a distance $x \gg R$ from the sheet. Find its speed when it passes through the center of the hole.

Part 3b: Find the speed of the particle as it passes through the center of the hole after being released from rest on L at a general distance x from the sheet.

[**Cavendish experiment**] – discussion from Morin, pp. 155-6

A dumbbell with two masses m on its ends hands from a very thin wire. The dumbbell is free to twist, although if it twists by an angle θ, the wire will provide a tiny restoring torque $\tau = -b\theta$ for some constant b. The dumbbell starts with no twist in the wire, and then two other masses M are placed (fixed) at the positions shown below.



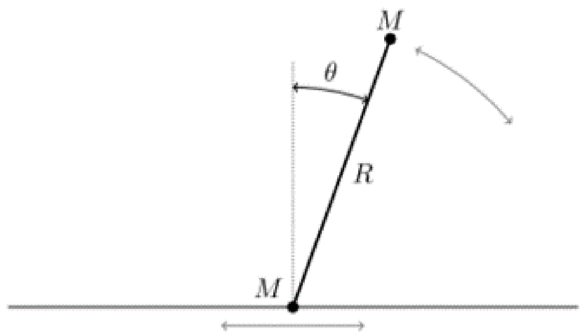
The masses produce attractive forces on the dumbbell masses and cause the dumbbell to twist counterclockwise. The dumbbell will oscillate back and forth before finally settling down at some tiny angle away from the initial position.

Let ℓ be half the length of the dumbbell and I be the moment of inertia of the dumbbell.

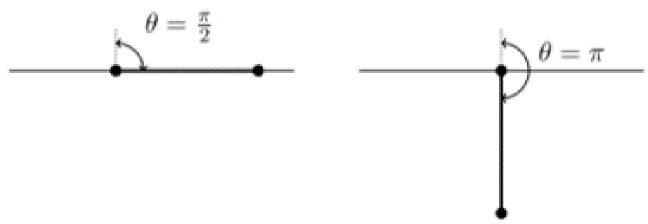
Given the period T of oscillations, the separation d between the centers of the masses in each pair and the final angular position θ of the dumbbell, obtain an estimate of the universal gravitational constant G in terms of m, M, ℓ, I, T, d and θ.

[USAPhO, 2019 B3]

A bead is placed on a horizontal rail, along which it can slide frictionlessly. It is attached to the end of a rigid, massless rod of length R. A ball is attached at the other end. Both the bead and the ball have mass M. The system is initially stationary, with the ball directly above the bead. The ball is then given an infinitesimal push, *parallel* to the rail.



Assume that the rod and ball are designed in such a way (not shown explicitly in the diagram) so that they can pass through the rail without hitting it. In other words, the rail only constrains the motion of the bead. Two subsequent states of the system are shown below.



a. Derive an expression for the force in the rod when it is horizontal, as shown at left above, and indicate whether it is tension or compression.

b. Derive an expression for the force in the rod when the ball is directly below the bead, as shown at right above, and indicate whether it is tension or compression.

c. Let θ be the angle the rod makes with the vertical, so that the rod begins at $\theta = 0$. Find the angular velocity $\omega = d\theta/dt$ as a function of θ.