Harvard Physics Circle

Lecture 11: Hydrostatics

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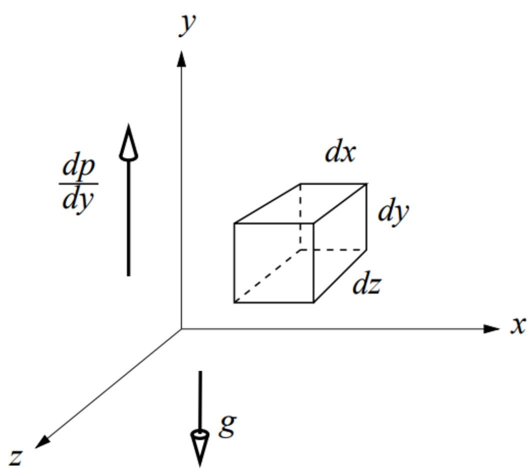
1 Theory

1.1 Hydrostatic Equation

Let us study the free body diagram of a fluid element at rest under gravity in the vertical y direction.



The only forces acting on this fluid element are gravity and pressure. Since it is at rest they have to cancel to zero.

$$\vec{F}_G + \vec{F}_P = 0 \tag{1}$$

We see that the from both sides must be equal in the x and z dimensions. For the y dimension we can write,

$$p(y)\,dx\,dz - p(y+dy)\,dx\,dz - \rho g\,dx\,dy\,dz = 0. \tag{2}$$

Since dy is an infinitesimally small length we can write,

$$p(y+dy) = p(y) + \frac{dp}{dy}dy. \tag{3}$$

Inserting this to the force balance equation, we get the differential form of the *Hydrostatic Equation.*

$$\boxed{\frac{dp}{dy} = -\rho g} \tag{4}$$

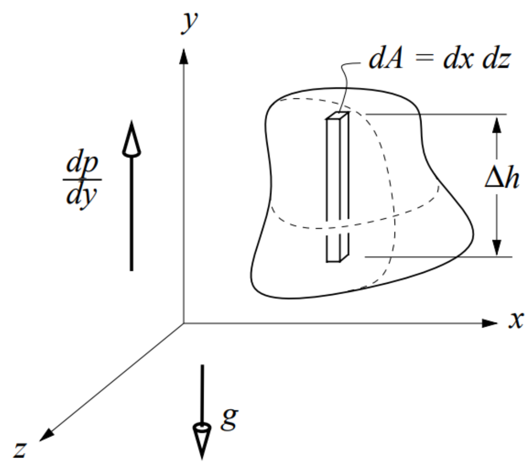
If the fluid is constant density, by integrating both sides, we get the integral form.

$$\boxed{p(y) = p_0 - \rho g y} \tag{5}$$

where p_0 is the pressure at $y = 0$.

1.2 Buoyancy

Now let us consider a solid object with an arbitrary shape, immersed in a fluid with constant density. We divide the volume into vertical volume elements.



We want to calculate the total force due to the pressure difference between the top surface and the bottom surface which is called *buoyancy*. We can write it for the small volume element that we chose,

$$dF = p(y)\,dx\,dz - p(y + \Delta h)\,dx\,dz. \tag{6}$$

We can use the hydrostatic equation to write $p(y + \Delta h) = p(y) - \rho g \Delta h$, which we can insert into our force calculation.

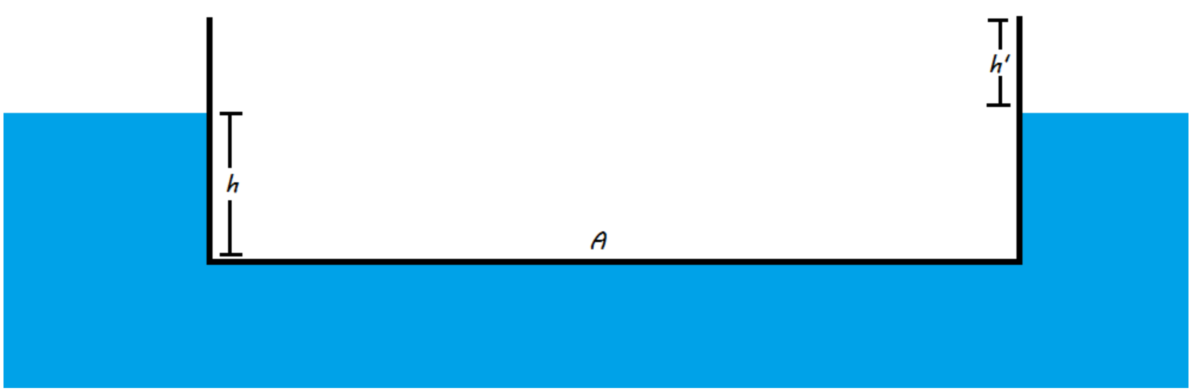
$$dF = \rho g \Delta h\,dx\,dz = \rho g\,dV \tag{7}$$

We see that the buoyancy force is proportional to the volume of the small volume element. To get the total force we have to sum for each volume element. We obtain,

$$\boxed{F = \rho g V}. \tag{8}$$

This is the mathematical form of the famous *Archimedes Principle.*

Note: when an object is partially immersed in a fluid, the buoyancy force is calculated with considering the immersed volume of the object.



So the buoyancy force here is $F = \rho g h A$.

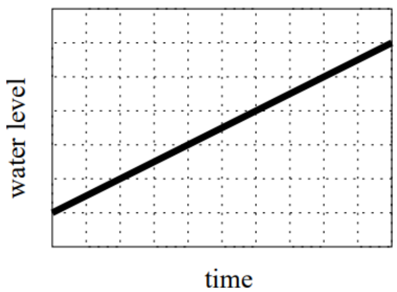
2 F=ma Problems

2.1 F=ma 2011

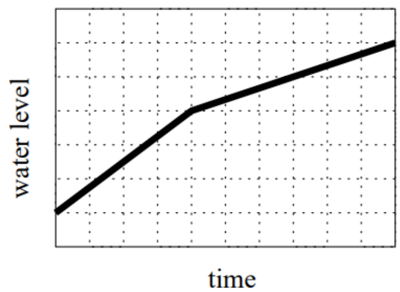
8. When a block of wood with a weight of 30 N is completely submerged under water the buoyant force on the block of wood from the water is 50 N. When the block is released it floats at the surface. What fraction of the block will then be visible above the surface of the water when the block is floating?

   (A) 1/15
   (B) 1/5
   (C) 1/3
   (D) 2/5
   (E) 3/5

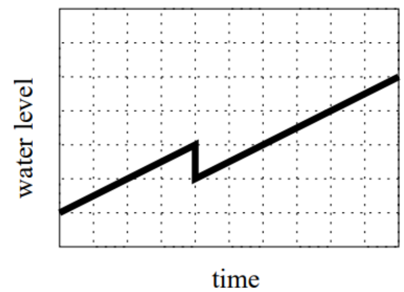
11. A large metal cylindrical cup floats in a rectangular tub half-filled with water. The tap is placed over the cup and turned on, releasing water at a constant rate. Eventually the cup sinks to the bottom and is completely submerged. Which of the following five graphs could represent the water level in the sink as a function of time?



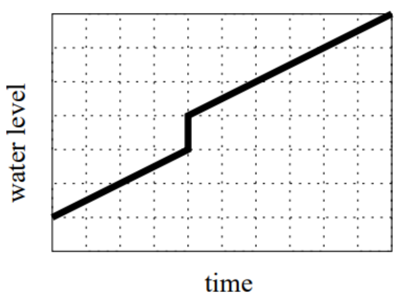
(A)



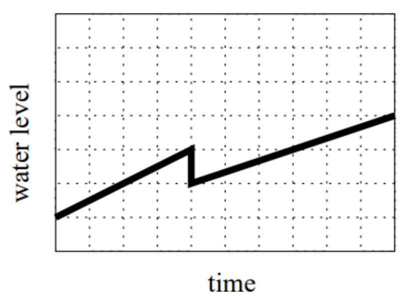
(B)



(C)



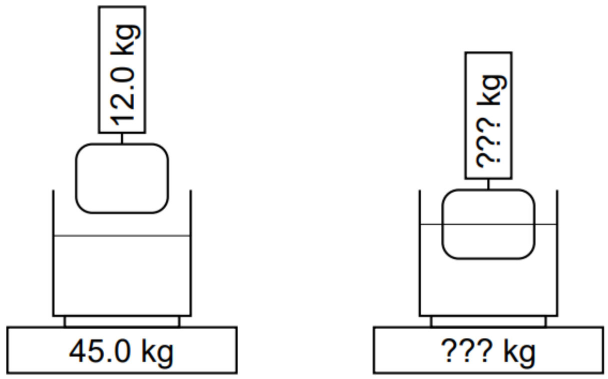
(D)



(E)

2.2 F=ma 2012

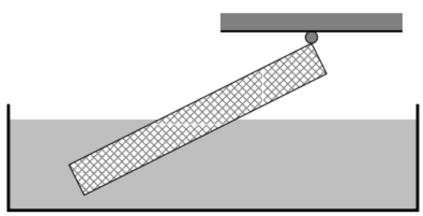
20. A container of water is sitting on a scale. Originally, the scale reads $M_1 = 45$ kg. A block of wood is suspended from a second scale; originally the scale read $M_2 = 12$ kg. The density of wood is 0.60 g/cm^3; the density of the water is 1.00 g/cm^3. The block of wood is lowered into the water until half of the block is beneath the surface. What is the resulting reading on the scales?



(A) $M_1 = 45$ kg and $M_2 = 2$ kg.
(B) $M_1 = 45$ kg and $M_2 = 6$ kg.
(C) $M_1 = 45$ kg and $M_2 = 10$ kg.
(D) $M_1 = 55$ kg and $M_2 = 6$ kg.
(E) $M_1 = 55$ kg and $M_2 = 2$ kg.

2.3 F=ma 2013

15. A uniform rod is partially in water with one end suspended, as shown in figure. The density of the rod is 5/9 that of water. At equilibrium, what portion of the rod is above water?



(A) 0.25
(B) 0.33
(C) 0.5
(D) 0.67
(E) 0.75

2.4 F=ma 2015

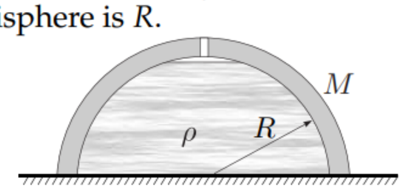
A U-tube manometer consists of a uniform diameter cylindrical tube that is bent into a U shape. It is originally filled with water that has a density ρ_w. The *total* length of the column of water is L. Ignore surface tension and viscosity.

20. Oil with a density half that of water is added to one side of the tube until the total length of oil is equal to the total length of water. Determine the equilibrium height difference between the two sides

    (A) L
    (B) $L/2$
    (C) $L/3$
    (D) $3L/4$
    (E) $L/4$

3 More Challenging Problems

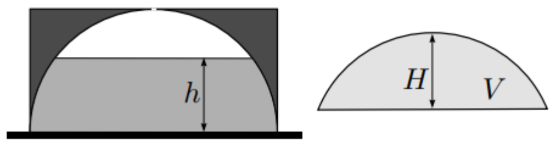
PROB 15. A hemispherical container is placed upside down on a smooth horizontal surface. Through a small hole at the bottom of the container, water is then poured in. Exactly when the container gets full, water starts leaking from between the table and the edge of the container. Find the mass of the container if water has density ρ and radius of the hemisphere is R.



IDEA 21: If water starts flowing out from under an upside down container, normal force must have vanished between the table and the edge of the container. Therefore force acting on the system container+liquid from the table is equal solely to force from hydrostatic pressure.

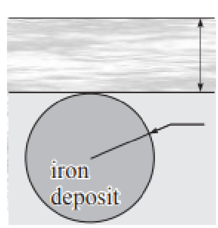
The latter is given by pS, where p is pressure of the liquid near the tabletop and S is area of the container's open side.

PROB 60. A vessel in the shape of a cylinder, whose height equals its radius R and whose cavity is half-spherical, is filled to the brim with water, turned upside down and positioned on a horizontal surface. The radius of the half-spherical cavity is also R and there is a little hole in the vessel's bottom. From below the edges of the freely lying vessel some water leaks out. How high will the remaining layer of water be, if the mass of the vessel is m and the water density is ρ? If necessary, use the formula for the volume of a slice of a sphere (see Fig.): $V = \pi H^2(R - H/3)$.



PROB 14. If a beam with square cross-section and very low density is placed in water, it will turn one pair of its long opposite faces horizontal. This orientation, however, becomes unstable as we increase its density. Find the critical density when this transition occurs. The density of water is $\rho_v = 1000\,\mathrm{kg/m^3}$.

PROB 17. Let us investigate the extent to which an iron deposit can influence water level. Consider an iron deposit at the bottom of the ocean at depth $h = 2\,\mathrm{km}$. To simplify our analysis, let us assume that it is a spherical volume with radius $1\,\mathrm{km}$ with density greater from the surrounding rock by $\Delta\rho = 1000\,\mathrm{kg/m^3}$. Presume that this sphere touches the bottom of the ocean with its top, i.e. that its centre is situated at depth $r + h$. By how much is the water level directly above the iron deposit different from the average water level?



4 References

- https://www.aapt.org/physicsteam/2018/exams.cfm

- http://www.physics.mcgill.ca/~keshav/230/olympiad.pdf