

Harvard Physics Circle

Lecture 14: Magnetism, Biot-Savart, Ampere's Law

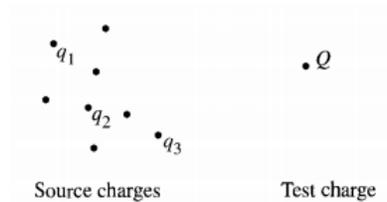
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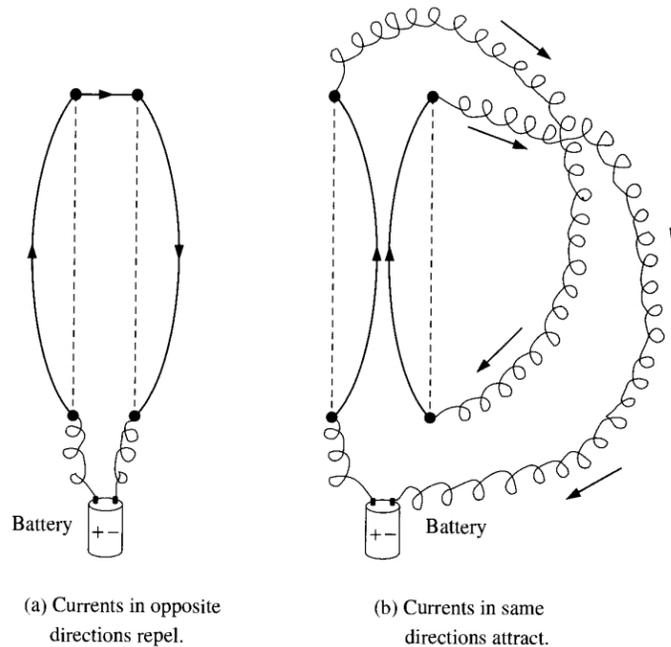
1 Theory

1.1 Magnetic Fields

We are dealing with the same problem of how charged particles interact with each other. We have a group of source charges and a test charge that moves under the influence of these source charges. Unlike electrostatics, however, the source charges are in motion.



One of the simplest experiments one can do to gain insight on how magnetism works is observing two parallel wires that have currents flowing through them.



The force causing this attraction and repulsion is not electrostatic since the wires are neutral. Even if they were not neutral, flipping the wires would not flip the direction of the force as we see in the experiment. Magnetic fields are what is responsible for this phenomenon. A stationary charge produces only an electric field \vec{E} around it, while a moving charge creates a magnetic field \vec{B} . We will first study the force acting on a charge under the influence of an ambient magnetic field, before we delve into how moving charges generate such magnetic fields.

1.2 The Lorentz Force Law

For a particle with charge q moving with velocity \vec{v} in a magnetic field \vec{B} , the force acting on the particle by the magnetic field is given by,

$$\vec{F} = q(\vec{v} \times \vec{B}). \quad (1)$$

This is known as the *Lorentz force law*. Just like $F = ma$, this law is based on experiments rather than being derived. Notice that unlike the electrostatic version of this law where the force is parallel to the electric field ($\vec{F} = q\vec{E}$), here, the force is perpendicular to both the velocity of the particle and the magnetic field. This property causes some interesting phenomena that are worth diving into.

Right Hand Rule is explained best visually.

Cyclotron Motion: Suppose we have a particle with mass m and charge q in an constant ambient magnetic field \vec{B} in the \hat{z} direction. Let the initial velocity of the particle be \vec{v} . Let us solve for the particle's motion. We have,

$$\vec{F} = q(\vec{v} \times \vec{B}) \Rightarrow m\dot{\vec{v}} = qB(\vec{v} \times \hat{z}). \quad (2)$$

Since this is a vector equation we can write 3 equations for each of the components.

$$m\dot{v}_x = qBv_y, \quad m\dot{v}_y = -qBv_x, \quad m\dot{v}_z = 0. \quad (3)$$

The third equation tells us that $v_z = \text{const.}$ The first two equations are solved by,

$$v_x = A \cos(\omega t + \phi), \quad v_y = -A \sin(\omega t + \phi). \quad (4)$$

where $\omega \equiv qB/m$ is the *cyclotron frequency*. A and ϕ are determined from the initial velocity of the particle. In the $x - y$ plane, the particle is undergoing uniform circular motion with angular frequency ω . We can find the radius of the motion by writing,

$$qvB = m \frac{v^2}{r} \Rightarrow r = \frac{mv}{qB} = \frac{v}{\omega}. \quad (5)$$

When we add the \hat{z} component of the particle's velocity, we see that the general 3D motion follows a helix. This is a very useful experimental technique to determine the charge and momenta of particles.

Another very important fact that results from the mathematical form of the Lorentz force law is that **magnetic forces do no work!** We can see this mathematically. Suppose the particle moves an infinitesimal distance $d\vec{l}$. The work done by the magnetic force is then,

$$\delta W = \vec{F} \cdot d\vec{l} = q(\vec{v} \times \vec{B}) \cdot d\vec{l} = q dt(\vec{v} \times \vec{B}) \cdot \vec{v} = 0 \quad (6)$$

since \vec{v} is perpendicular to $\vec{v} \times \vec{B}$. There are confusing circumstances where magnetic forces appear to be doing work, such as one magnet lifting another. However, there is always another source where the work is done by.

Problem 5.3 In 1897 J. J. Thomson “discovered” the electron by measuring the charge-to-mass ratio of “cathode rays” (actually, streams of electrons, with charge q and mass m) as follows:

(a) First he passed the beam through uniform crossed electric and magnetic fields \mathbf{E} and \mathbf{B} (mutually perpendicular, and both of them perpendicular to the beam), and adjusted the electric field until he got zero deflection. What, then, was the speed of the particles (in terms of E and B)?

(b) Then he turned off the electric field, and measured the radius of curvature, R , of the beam as deflected by the magnetic field alone. In terms of E , B , and R , what is the charge-to-mass ratio (q/m) of the particles?

It is useful to also study the Lorentz force law when it is applied to currents flowing through wires. Since current is defined as charge flowing per unit time, we can relate charge and velocity to current and length of wire. Suppose we have a wire with a current I running through it. If we write the line density of charges (charge per unit length of wire) running through the wire as λ . We can write the current as a vector. Let us mark a point on the wire and count all the charges that pass through in time δt . By definition we can write the total charge as $\delta Q = I\delta t$. We can also write the total charge as $\delta Q = \lambda\delta l$, where δl is $v\delta t$. So we have,

$$I = \lambda v, \quad (7)$$

which we can also write in vector form,

$$\vec{I} = \lambda \vec{v}. \quad (8)$$

An important note here is that the charges that contribute to λ are only the moving charges that form the current. Almost always it is the negatively charged electrons that are moving in materials that form the current. Therefore, you can think of λ as the electron charge line density.

So now we can write the magnetic force on a segment of a wire as,

$$\vec{F} = \int dq (\vec{v} \times \vec{B}) = \int dl \lambda (\vec{v} \times \vec{B}) = \int dl (\vec{I} \times \vec{B}). \quad (9)$$

It is usually the case that \vec{I} and dl point in the same direction and I is constant along the wire. So we can write,

$$\vec{F} = I \int d\vec{l} \times \vec{B} \quad (10)$$

Magnetic Crane: Suppose we have a rectangular loop of wire with width a that carries mass m . The top of the wire is immersed in a uniform magnetic field \vec{B} that points into the page. What is the current I needed to cancel the weight of the mass?

We see with the right hand rule that the current needed must circulate clockwise so that $\vec{I} \times \vec{B}$ points upwards. The integral of a constant is simple to evaluate and gives,

$$F = IBa \quad \Rightarrow \quad mg = IBa \quad (11)$$

So the needed current is,

$$I = \frac{mg}{Ba} \quad (12)$$

Exercise: If I is larger than this value, the weight will be lifted upwards. Who is doing the work of lifting the weight?

Exercise: We have derived the version of the Lorentz force law for 1D line currents. Derive the mathematical form of the Lorentz force for 2D surface current densities and 3D volume current densities.

1.3 Biot-Savart Law

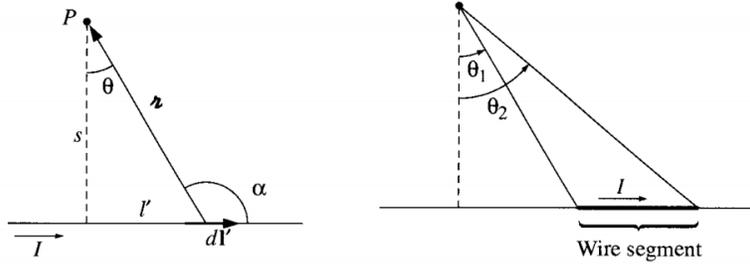
We will now study magnetic fields that are generated by steady currents which is called *magnetostatics*. This is analogous to electrostatics which deals with electric fields generated by stationary charges. An important note is that a moving point charge is not a steady current. A steady current is defined such that at every point in space, the current is constant over time. For a steady current that flows through a line, the magnetic field it generates is given by the *Biot-Savart law*:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int dl' \frac{\vec{I} \times \hat{\mathbf{r}}}{r^2}. \quad (13)$$

where μ_0 is called *permeability of free space* and its value is,

$$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \quad (14)$$

Magnetic Field of an Infinite Wire: Let us calculate the magnetic field at distance s away from an infinite wire carrying current I using the Biot-Savart law.



We can start with,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d\vec{l}' \frac{\vec{I} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{r}}{r^2} \quad (15)$$

We see from the figure that $d\vec{l}' \times \hat{r}$ points out of the page and has magnitude,

$$dl' \cos \theta. \quad (16)$$

We can express this in terms of s by differentiating $l' = s \tan \theta$. We get,

$$dl' = \frac{s}{\cos^2 \theta} d\theta. \quad (17)$$

We can also write,

$$r^2 = \frac{s^2}{\cos^2 \theta}. \quad (18)$$

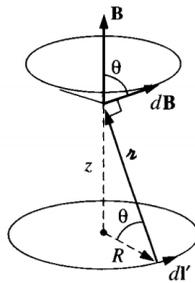
Putting everything together, we get the following integral to evaluate,

$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} d\theta \left(\frac{\cos^2 \theta}{s^2} \right) \left(\frac{s}{\cos^2 \theta} \right) \cos \theta = \frac{\mu_0 I}{4\pi s} [\sin \theta_2 - \sin \theta_1] \quad (19)$$

So for an infinite wire we get, $\theta_1 = -\pi/2$, $\theta_2 = \pi/2$.

$$B = \frac{\mu_0 I}{2\pi s} \quad (20)$$

Magnetic Field of a Ring Current on its Axis Let us calculate the magnetic field of a circular current above its center.



We see that the horizontal components cancel. So we can sum up the vertical component of $d\vec{B}$ to get the final magnetic field, which will be purely in the vertical direction.

$$B(z) = \frac{\mu_0 I}{4\pi} \int dl' \frac{\cos \theta}{r^2} \quad (21)$$

This is a constant integral which is trivial to evaluate $\int dl' = 2\pi R$. So we get,

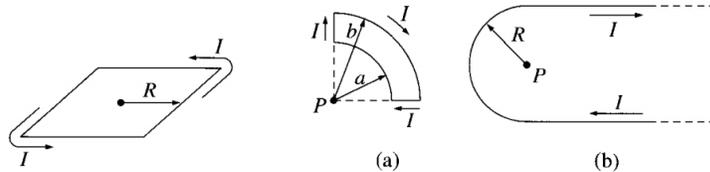
$$B(z) = \frac{\mu_0 I \cos \theta}{4\pi r^2} 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \quad (22)$$

Exercise: Derive the Biot-Savart law for surface and volume currents.

Problem 5.8

- (a) Find the magnetic field at the center of a square loop, which carries a steady current I . Let R be the distance from center to side (Fig. 5.22).
- (b) Find the field at the center of a regular n -sided polygon, carrying a steady current I . Again, let R be the distance from the center to any side.
- (c) Check that your formula reduces to the field at the center of a circular loop, in the limit $n \rightarrow \infty$.

Problem 5.9 Find the magnetic field at point P for each of the steady current configurations shown in Fig. 5.23.



1.4 Ampere's Law

Just like Gauss' law in electrostatics where the integral of electric flux over a closed surface is proportional to the total enclosed charge, the loop integral of magnetic fields is proportional to the enclosed current, which is called *Ampere's law*. Mathematically it is stated as,

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I_{\text{enc}}. \quad (23)$$

Magnetic Field of the Infinite Wire with Ampere's Law Let us solve the earlier problem, this time using Ampere's law. We choose a symmetric circular Amperian loop centered around the wire with radius s . Since the system has cylindrical symmetry, and right hand rule applies, we know that the magnetic field is perpendicular to the Amperian loop and has uniform strength over the loop. So we can write,

$$\oint d\vec{l} \cdot \vec{B} = B \oint dl = B2\pi s = \mu_0 I \quad (24)$$

which gives,

$$B = \frac{\mu_0 I}{2\pi s} \quad (25)$$

obtained with significantly less effort!

Magnetic Field of an Infinite Uniform Surface Current $\vec{K} = K\hat{x}$ We have an infinite surface current that spans the $x - y$ plane and flows in the \hat{x} direction. Let us find the magnetic field at a point above the surface by height h .

From looking at Biot-Savart law we know that the magnetic field cannot have an \hat{x} component since all infinitesimal fields we add have to be perpendicular to \hat{x} . It cannot have a \hat{z} component either (**Exercise:** you can see this by flipping it in various directions and using the symmetry arguments). So it only has a \hat{y} component. So we pick a rectangular amperian loop and apply Ampere's law. We get,

$$\oint d\vec{l} \cdot \vec{B} = 2Bl = \mu_0 I_{\text{enc}} = \mu_0 Kl \quad (26)$$

which gives,

$$B = \frac{\mu_0 K}{2} \quad (27)$$

and its direction is given by right hand rule.

Magnetic Field of a Long Solenoid A very common tool to create confined magnetic fields is obtained by wrapping current carrying a wire around a cylinder. The resulting shape is called *solenoid*. If the wire is thin enough, this approximates a cylindrical uniform surface current $K = nI$ where n is the number of windings per unit length and I is the current flowing through the wire.

Using symmetry arguments, we can eliminate the possibility of having radial and azimuthal components (**Exercise**). So the magnetic field is parallel to the axis. We can construct two square Amperian loops with side length L . The first one is purely on the outside of the solenoid with two sides parallel to the axis and the remaining sides in the radial direction. Let the inner side be distance a away from the central axis and the outer be b away. We can write Ampere's law and obtain,

$$\oint d\vec{l} \cdot \vec{B} = [B(a) - B(b)]L = \mu_0 I_{\text{enc}} = 0 \quad \Rightarrow \quad B(a) = B(b) \quad (28)$$

Since it does not depend on the distance it must be zero everywhere since it has to vanish at large distance.

Let the second Amperian loop be almost the same orientation the first one but dipped into the solenoid with $a < R$ and $b > R$ where R is the radius of the solenoid. We can write Ampere's law for this case as,

$$\oint d\vec{l} \cdot \vec{B} = B(a)L = \mu_0 I_{\text{enc}} = \mu_0 nIL. \quad (29)$$

Which again is independent of a . So we have a constant magnetic field inside and outside of the solenoid. Outside, it is constant at $\vec{0}$. Inside it is $\vec{B} = \mu_0 nI\hat{z}$. Solenoids are parallel plate capacitors of magnetostatics in the sense that they produce a confined uniform magnetic field.

Exercise: Calculate the magnetic field of the long solenoid using Biot-Savart law.

1.5 Magnetostatic Boundary Conditions

Before we go into the boundary conditions, we should state a very important property of magnetic fields. **Magnetic fields have zero flux over a closed surface!** Since Gauss' law states that the flux over a closed surface gives the net enclosed charge, we conclude that there are no magnetic charges. (active area of research). This is written mathematically as,

$$\oint_S d\vec{a} \cdot \vec{B} = 0, \quad \forall \text{closed surface } S. \quad (30)$$

Just like when an electric field has a discontinuity wherever there is a surface charge, a magnetic field gets a discontinuity when there is a surface current. Suppose we have a thin surface that has a surface current \vec{K} flowing through it. By picking a Gaussian surface that is shaped like a thin wafer we can find that the perpendicular components of the magnetic field right above and right below the surface have to be equal.

$$B_{\text{above}}^{\perp} = B_{\text{below}}^{\perp} \quad (31)$$

The tangential components can be related by constructing an Amperian loop perpendicular to the current \vec{K} . We get,

$$\oint d\vec{l} \cdot \vec{B} = (B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel})l = \mu_0 Kl \quad (32)$$

We can put together these two boundary condition equations into a single vector equation as,

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 (\vec{K} \times \hat{n}) \quad (33)$$

2 More Problems

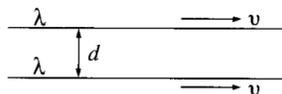


Figure 5.26

Problem 5.12 Suppose you have two infinite straight line charges λ , a distance d apart, moving along at a constant speed v (Fig. 5.26). How great would v have to be in order for the magnetic attraction to balance the electrical repulsion? Work out the actual number. . . Is this a reasonable sort of speed?⁸

Problem 5.13 A steady current I flows down a long cylindrical wire of radius a (Fig. 5.40). Find the magnetic field, both inside and outside the wire, if

- The current is uniformly distributed over the outside surface of the wire.
- The current is distributed in such a way that J is proportional to s , the distance from the axis.

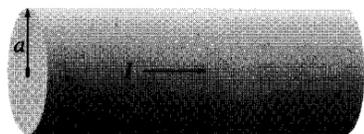


Figure 5.40

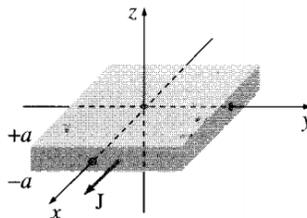


Figure 5.41

Problem 5.14 A thick slab extending from $z = -a$ to $z = +a$ carries a uniform volume current $\mathbf{J} = J \hat{\mathbf{x}}$ (Fig. 5.41). Find the magnetic field, as a function of z , both inside and outside the slab.

3 References

- David J. Griffiths, *Introduction to Electrodynamics*