§1 Theory

(Momentum conservation law/generalized Newton’s 2nd law.) For the net momentum $\vec{p} = \sum_i m_i \vec{v}_i$ of a system of point masses $m_i$ upon which a total exterior force $\vec{F}$ acts, we have:

$$\frac{d\vec{p}}{dt} = \vec{F}$$

This implies that the net momentum is conserved if $\vec{F}$.

Center of mass For a system of point masses $m_i$, each at coordinate $\vec{r}_i$, we have that the coordinate of the center of mass is:

$$\vec{r}_c = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

Corollaries:

- If external forces cancel out on some direction $i$, then the center of mass will be stationary on the coordinate $i$.
- For a non-homogeneous body of density $\rho(x, y, z)$, we have the integral form that defines its center of mass:

$$\vec{r}_c = \frac{\int \vec{r} dm}{\int dm}$$

- Newton’s third law If two bodies have interacted, then the changes of momenta of the two bodies are equal and opposite.

§2 Warm-up problems

§2.1 F=ma, 2018

A 3.0 kg mass moving at 30 m/s to the right collides elastically with a 2.0 kg mass traveling at 20 m/s to the left. After the collision, the center of mass of the system is moving at a speed of

(A) 5 m/s
(B) 10 m/s
(C) 20 m/s
(D) 24 m/s
(E) 26 m/s
§2.2 F=ma, 2015

An object launched vertically upward from the ground with a speed of 50 m/s bounces off of the ground on the return trip with a coefficient of restitution given by $C_R = 0.9$, meaning that immediately after a bounce the upward speed is 90% of the previous downward speed. The ball continues to bounce like this; what is the total amount of time between when the ball is launched and when it finally comes to a rest? Assume the collision time is zero; the bounce is instantaneous. Treat the problem as ideally classical and ignore any quantum effects that might happen for very small bounces.

(A) 71 s  
(B) 100 s  
(C) 141 s  
(D) 1000 s  
(E) $\infty$ (the ball never comes to a rest)

§2.3 F=ma, 2017

A particle of mass $m$ moving at speed $v_0$ collides with a particle of mass $M$ which is originally at rest. The fractional momentum transfer $f$ is the absolute value of the final momentum of $M$ divided by the initial momentum of $m$.

If the collision is completely inelastic, under what condition will the fractional momentum transfer between the two objects be a maximum?

(A) $m/M \ll 1$  
(B) $0.5 < m/M < 1$  
(C) $m = M$  
(D) $1 < m/M < 2$  
(E) $m/M \gg 1$

If the collision is perfectly elastic, what is the maximum possible fractional momentum transfer, $f_{\text{max}}$?

(A) $0 < f_{\text{max}} < \frac{1}{2}$  
(B) $f_{\text{max}} = \frac{1}{2}$  
(C) $\frac{1}{2} < f_{\text{max}} < \frac{3}{2}$  
(D) $f_{\text{max}} = 2$  
(E) $f_{\text{max}} \geq 3$

The fractional energy transfer is the absolute value of the final kinetic energy of $M$ divided by the initial kinetic energy of $m$.

If the collision is perfectly elastic, under what condition will the fractional energy transfer between the two objects be a maximum?

(A) $m/M \ll 1$  
(B) $0.5 < m/M < 1$  
(C) $m = M$  
(D) $1 < m/M < 2$  
(E) $m/M \gg 1$
§2.4 $F=ma$, 2016

An object of mass $m_1$ initially moving at speed $v_0$ collides with an originally stationary object of mass $m_2 = \alpha m_1$, where $\alpha < 1$. The collision could be completely elastic, completely inelastic, or partially inelastic. After the collision the two objects move at speeds $v_1$ and $v_2$. Assume that the collision is one dimensional, and that object one cannot pass through object two.

After the collision, the speed ratio $r_2 = v_2/v_0$ of object 2 is bounded by

(A) $(1 - \alpha)/(1 + \alpha) \leq r_2 \leq 1$
(B) $(1 - \alpha)/(1 + \alpha) \leq r_2 \leq 1/(1 + \alpha)$
(C) $\alpha/(1 + \alpha) \leq r_2 \leq 1$
(D) $0 \leq r_2 \leq 2\alpha/(1 + \alpha)$
(E) $1/(1 + \alpha) \leq r_2 \leq 2/(1 + \alpha)$

§2.5 $F=ma$, 2019

A block of mass $m$ is launched horizontally onto a curved wedge of mass $M$ at a velocity $v$. What is the maximum height reached by the block after it shoots off the vertical segment of the wedge? Assume all surfaces are frictionless; both the block and the curved wedge are free to move. The curved wedge does not tilt or topple.

![Diagram of a block and a curved wedge](image)

(A) $\frac{v^2}{2g}$
(B) $\left(\frac{m}{m+M}\right)^2 \cdot \frac{v^2}{2g}$
(C) $\left(\frac{M}{m+M}\right)^2 \cdot \frac{v^2}{2g}$
(D) $\frac{m}{m+M} \cdot \frac{v^2}{2g}$
(E) $\frac{M}{m+M} \cdot \frac{v^2}{2g}$
§3 Harder problems

§3.1 Estonian-Finnish Olympiad, 2012

Consider a perfectly elastic collision of two balls, one of which has mass $M$ and is moving with velocity $v$. The other has mass $m \leq M$ and stays initially at rest. The collision is not necessarily central. The surface of the balls is slippery, so the balls will not rotate.

i. (1 pt) What are the momenta of the balls before the collision in the frame of reference where the centre of mass of the whole system stays at rest?

ii. (3 pts) What are the momenta of the balls (by moduli) after the collision in the mass centre’s frame of reference?

iii. (3 pts) What is the maximal angle $\alpha$ by which the trajectory of the initially moving ball can be inclined as a result of the collision?

§3.2 Estonian-Finnish Olympiad, 2012

Three little cylinders are connected with weightless rods, where there is a hinge near the middle cylinder, so that the angle between the rods can change freely. Initially this angle is a right angle. Two of the cylinders have mass $m$, another one at the side has the mass $4m$. Find the acceleration of the heavier cylinder immediately after the motion begins. Ignore friction.

§3.3 Estonian-Finnish Olympiad, 2014

4. SUPERBALLS (5 points) $n + 1$ elastic balls are dropped so that they are exactly above each other, with a very small gap between each. Bottom ball has a mass of $m_0$, the one above has a mass of $f m_0$, next $f^2 m_0$ and so on, until the topmost ball with mass $f^n m_0$, where $f < 1$. At the moment when the bottom ball touches the ground, all the balls are moving with the speed $v$.

i) (1 point) After the collision between the two bottommost balls, what is the speed $v_1$ of the second ball from the bottom?

ii) (3 points) What is the speed of the topmost ball $v_n$ after all collisions?

iii) (1 point) How many times higher would that ball fly compared to the initial drop height $h$? Take $f = 0.5$ and $n = 10$.

It maybe useful that that sequence $a_0 = 1$, $a_{k+1} = a_k \alpha + \beta$ has a general term $a_n = a^n + \beta a^{n-1}$, where $\alpha$ and $\beta$ are constants.
§4 Homework/Additional problems

Most problems that involve momentum also involve its angular equivalent, the angular momentum which we will discuss in some future meeting. I would suggest trying to do the problems regarding momentum in Morin’s book (Morin, David J. Problems and Solutions in Introductory Mechanics) and also check out Kalda’s book (first reference below).

§5 References