Simple Harmonic Motion, Harvard Physics Circle

Gopal K. Goel

November 4, 2020

Review

In the classic spring example, a particle of mass m is subject to a force $F(x) = -kx$, and we want to find its trajectory $x(t)$. Letting $\omega = \sqrt{k/m}$, the $F = ma$ equation becomes

$$\ddot{x} + \omega^2 x = 0.$$

The solutions to this can be parameterized as

$$x(t) = A\cos(\omega t + \phi),$$

where A and ϕ are constants that depend on the initial conditions, or as

$$x(t) = B\cos(\omega t) + C\sin(\omega t),$$

where B and C are constants that depend on the initial conditions. A consequence of these solutions for $x(t)$ is that the period of the motion of the particle is

$$T = 2\pi/\omega = 2\pi\sqrt{\frac{m}{k}},$$

and that, as expected, the energy

$$E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$$

is conserved.

Another classic example is the pendulum, where we have a mass m hanging from a string of length ℓ. Letting θ be the angle of the string from the vertical, the torque equation reads

$$-mg\ell\sin\theta = (m\ell^2)\ddot{\theta}.$$

By the small angle approximation $\sin\theta \approx \theta$, this becomes

$$\ddot{\theta} + \omega^2\theta = 0,$$

where $\omega = \sqrt{g/\ell}$, so the same analysis from before holds.

In general, suppose a system has a parameter p describing its state (such as position or angle in our previous two examples) where p has a stable equilibrium around 0. In most cases, the appropriate force or torque equation can be written after approximation as

$$\ddot{p} + \omega^2 p = 0$$

for some ω depending on the exact setup. This leads to oscillations of similar sinusoidal nature with period $2\pi/\omega$, as described prior.

Problems

Problem 1. [F=ma 2014/8] An object of mass M is hung on a vertical spring of spring constant k and is set into vertical oscillations. The period of this oscillation is T_0. The spring is then cut in half and the same mass is attached and the system is set up to oscillate on a frictionless inclined plane making an angle θ to the horizontal. Determine the period of the oscillations on the inclined plane in terms of T_0.

(A) T_0

(B) $T_0/2$

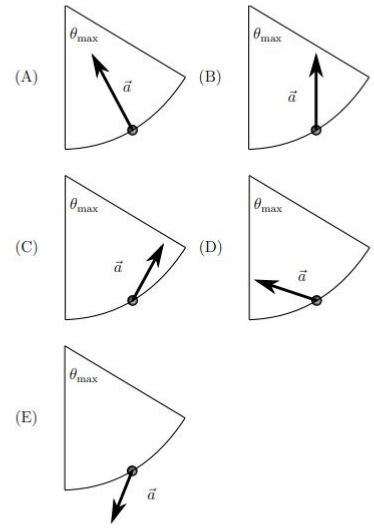
(C) $2T_0 \sin\theta$

(D) $T_0/\sqrt{2}$

(E) $T_0 \sin\theta/\sqrt{2}$

Problem 2. [F=ma 2015/13] A pendulum consists of a small bob of mass m attached to a fixed point by a string of length L. The pendulum bob swings down from rest from an initial angle $\theta_{\text{max}} < 90$ degrees.

Consider the pendulum bob when it is at an angle $\theta = \frac{1}{2}\theta_{\text{max}}$ on the way up. What is the direction of the acceleration vector?



Problem 3. [F=ma 2016/14 and 2016/15] See the end of this document for problem statements.

Problem 4. [F=ma 2017/2] A mass m hangs from a massless spring connected to the roof of a box of mass M. When the box is held stationary, the mass-spring system oscillates vertically with angular frequency ω. If the box is dropped and falls freely under gravity, how will the angular frequency change?

(A) ω will be unchanged

(B) ω will increase

(C) ω will decrease

(D) Oscillations are impossible under these conditions

(E) ω increase or decrease depending on the values of M and m

Problem 5. [F=ma 2018 A/11] A light, uniform, ideal spring is fixed at one end. If a mass is attached to the other end, the system oscillates with angular frequency ω. Now suppose the spring is fixed at the other end, then cut in half. The mass is attached between the two half springs.

The new angular frequency of oscillations is

(A) $\omega/2$

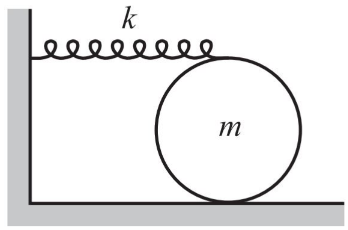
(B) ω

(C) $\sqrt{2}\omega$

(D) 2ω

(E) 4ω

Problem 6. [[M1] Problem 10.13]



A uniform solid cylinder with mass m and radius R is connected at its highest point to a spring (at its relaxed length) with spring constant k, as shown in the figure. If the cylinder rolls without slipping on the ground, what is the frequency of small oscillations?

Problem 7. [Classical] A mass m oscillates on a spring with spring constant k and amplitude A. Over a very long period of time (compared to the period of oscillation), the spring constant decays to half of its original value. What is the new amplitude of oscillation?

Problem 8. [[PPP] 77] A small bob of mass m is attached to two light, unstretched, identical springs. The springs are anchored at their far ends and arranged along a straight line. If the bob is displaced in a direction perpendicular to the line of the springs by a small length ℓ, the period of oscillation of the bob is $2T$. Find the period if the bob is displaced by length 2ℓ.

More Practice

- Do all the problems from [M1] chapter 10.
- Do USAPhO 2015 A3: https://www.aapt.org/physicsteam/2015/upload/E3-2-5.pdf
- Do IPhO 1992 Problem 2: https://www.ioc.ee/~kalda/ipho/23rd_IPhO_1992_Theo_Question_2.pdf

References

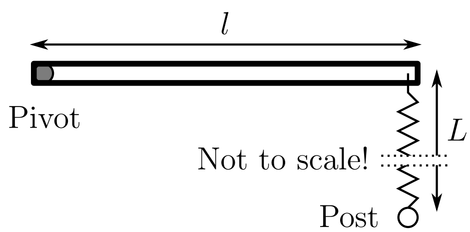
[M1] Morin, David J. *Problems and Solutions in Introductory Mechanics.*

[M2] Morin, David J. *Introduction to Classical Mechanics: With Problems and Solutions.*

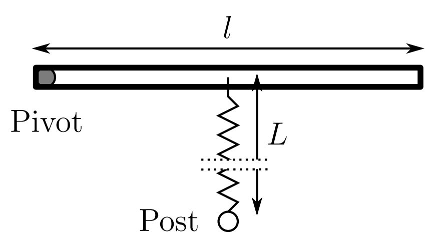
[PPP] Gnädig, Peter and Honyek Gyula and Riley, Ken. *200 Puzzling Physics Problems With Hints and Solutions.*

The following information applies to questions 14 and 15.

A uniform rod of length l lies on a frictionless horizontal surface. One end of the rod is attached to a pivot. An un-stretched spring of length $L \gg l$ lies on the surface perpendicular to the rod; one end of the spring is attached to the movable end of the rod, and the other end is attached to a fixed post. When the rod is rotated slightly about the pivot, it oscillates at frequency f.

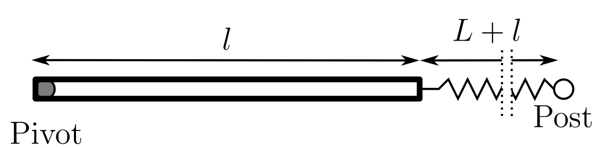


14. The spring attachment is moved to the midpoint of the rod, and the post is moved so the spring remains unstretched and perpendicular to the rod. The system is again set into small oscillations. What is the new frequency of oscillation?



(A) $f/2$
(B) $f/\sqrt{2}$
(C) f
(D) $\sqrt{2}f$
(E) $2f$

15. The spring attachment is moved back to the end of the rod; the post is moved so that it is in line with the rod and the pivot and the spring is unstretched. The post is then moved away from the pivot by an additional amount l. What is the new frequency of oscillation?



(A) $f/3$
(B) $f/\sqrt{3}$
(C) f
(D) $\sqrt{3}f$
(E) $3f$