Harvard Physics Circle

Relativistic Dynamics, Collisions, Problem-Solving Strategies

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§1 Relativistic Energy and Momentum

We define the rest energy of a particle of mass m to be $\boxed{E_0 = m_0c^2}$. When the particle moves with a velocity v, it's mass changes into $\boxed{m = m_0\gamma = \frac{m_0}{\sqrt{1-v^2/c^2}}}$ in the stationary reference frame. The energy of the particle is now

$$\boxed{E = mc^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} = E_0 + E_{kin}}$$

This is how we compute the kinetic energy of an object moving relativistically. The absolute value of the momentum is

$$\boxed{p = mv = \frac{m_0v}{\sqrt{1-v^2/c^2}}}$$

We can compute the force acting on a particle by

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m_0\frac{dv}{dt}\Big(\frac{\sqrt{1-v^2/c^2} + v^2/c^2\frac{1}{\sqrt{1-v^2/c^2}}}{1-v^2/c^2}\Big) = \boxed{\frac{m_0(dv/dt)}{(1-v^2/c^2)^{3/2}} = F}$$

Imagine we have a constant force $\vec{F}$ that acts on this particle. How would you find the velocity at some time t? An easy way is to simply integrate the momentum since F is constant:

$$p = F\cdot t \implies \frac{m_0v}{\sqrt{1-v^2/c^2}} = F\cdot t$$

Solving for v, we get:

$$v(t) = c\cdot\frac{1}{\sqrt{(m_0c/Ft)^2+1}}$$

Using these definitions for energy and momentum, the most important take out from this section is the formula which relates these quantities, namely:

$$\boxed{E^2 - p^2c^2 = E_0^2}$$

§2 Energy-Momentum Four-Vector

By looking at the formula from above, we notice that E_0 is constant because it is defined in the rest frame. This motivates the formalism of the energy/momentum four-vector of a particle since we have this invariant term. Since the way in which we write this four-vector is similar to the space-time coordinates four-vector from last time, I also include the latter here. **They are both defined so that the length of a four-vector is invariant under a coordinate transformation.**

$$s = (ct, x, y, z); \quad p = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

The length of four-vectors $x = (x_1, x_2, x_3, x_4)$ in relativity is defined as $\boxed{x^2 = x_1^2 - x_2^2 - x_3^2 - x_4^2}$. More formally, we can define the metric tensor (don't get scared, it's just a matrix):

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We compute the length of the four vector as:

$$x^2 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Note: Some people define η as being the negative of our η, which basically just changes the signs, but which doe snot affect the fact that the length of the four vector is invariant under frame changes.

§3 Conservation of Energy-Momentum

The conservation here refers to the invariance of the four-vector we discussed above. For the energy-momentum four-vector, we have in particular

$$E^2 - p^2c^2 = E_0^2 = \text{ constant}$$

§4 Relativistic Collision Problems

Most of the theory we talked about above is used in nuclear physics, in which we study the collision of different types of particle.

At CERN they use clusters of protons accelerated up to $E_{kin} = 7000GeV$. To accelerate them to this energy, they use a sequence of different accelerators, the last one begin LHC (Large Hadron Collider) which has a total length of $26659m$. The kinetic energies at the end of each stage are as follows:

Stage	Final kinetic energy of the protons
1	50 MeV
2	1,4 GeV
3	25 GeV
4	450 GeV
5	7000 GeV

We know the rest energy of a proton is $E_0 = 938$ MeV.

a) Find the ratio of the mass of a proton and its rest mass as a function of the kinetic energy.

b) Two clusters of protons with kinetic energy $E_{kin} = 7000 GeV$ collide linearly. What is the relative velocity of the protons in one cluster relative to the protons in the other one.

c) Find out in which stage of the acceleration process it become possible that from the collision of an accelerated proton with a resting proton we can get an anti-proton, according to the reaction: $p + p \to p + p + p + \bar{p}$. The anti-proton $\bar{p}$ has the same rest mass as a proton, and negative charge $-e$. Consider the minimum kinetic energy required.

§5 Loretz matrix transformations

We talked last session about relativistic cinematics, and in particular we introduced the following Lorentz transformations:

$$ct' = \gamma\Big(ct - xv/c\Big); \quad x' = \gamma(x - vt); \quad y' = y; \quad z' = z$$

We often write this using $\beta = v/c$ and the four-vectors $S' = (ct', x', y', z')$, $S = (ct, x, y, z)$:

$$X' = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} X$$

§6 Relativistic Doppler Effect

The change in frequency of a wave when viewed from a moving frame is called the Doppler Effect and we encounter it on a daily basis. Using the same technique as for the classical case, find the ratio of the observed frequency over the rest frame frequency as seen by an observer moving with velocity v. Assume you are given the wave frequency in the rest frame, and that the velocity v makes an angle θ with the lines uniting the wave origin and the moving observer.

§7 Accelerated relativistic frame

This is something we somewhat discussed above, when we defined relativistic momentum and deduce the formula for a force that acts on a relativistic particle. If you want to try out a more well-defined setup in which you can deduce relativistic accelerations, take a look at Problem 2, APhO 2013, 'Relativistic Correction on GPS Satellite'.

§8 Minkowski diagrams

This is something worth reading about, and which you can apply in Problem 2, APhO 2013, 'Relativistic Correction on GPS Satellite'. In particular, when working with simple systems (like all the ones you will encounter in high-school), it is debatable whether using these diagrams is time-efficient. Additionally, they definitely not safe-proof since one can easily get something wrong. However,as you can see in the problem mentioned above, they can be useful for things like satellites (which are rather complicated).

§9 Practice Problem for Collisions - Problem 3A, IPhO 2003, 'Neutrino mass and Neotron Decay'

A free neutron of mass m_n decays at rest in the laboratory frame of reference into three non-interacting particles: a proton, an electron, and an anti-neutrino. The rest mass of the proton is m_p, while the rest mass of the anti-neutrino m_ν is assumed to be nonzero and much smaller than the rest mass of the electron m_e. Denote the speed of light in vacuum by c. The measured values of mass are as follows: $m_n = 939.56563$ MeV/c, $m_p = 938.27231$ MeV/c, $m_e = 0.5109907$ MeV/c. In the following, all energies and velocities are referred to the laboratory frame. Let E be the total energy of the electron coming out of the decay.

Find the maximum possible value E_{max} of E and the speed v_m of the anti-neutrino when $E = E_{max}$. Both answers must be expressed in terms of the rest masses of the particles and the speed of light. Given that $m_\nu < 7.3$ eV/c, compute E_{max} and the ratio v_m/c to 3 significant digits.

§10 References

1. Problem 2, APhO 2013, 'Relativistic Correction on GPS Satellite'
   http://asianphysicsolympiad.org/papho2010.html#APhO2013

2. Problem 3A, IPhO 2003, 'Neutrino mass and Neotron Decay'
   Problem: https://www.ioc.ee/~kree/students/iphoTable/files/ipho/2003_Taiwan_p3.pdf
   Solution: https://www.ioc.ee/~kree/students/iphoTable/files/ipho/2003_Taiwan_p3Sol.pdf

3. Romanian National Physics Contest 'Evrika' 2012

4. Relativistic Doppler Effect
   https://en.wikipedia.org/wiki/Relativistic_Doppler_effect

5. Minkowski diagrams
   https://en.wikipedia.org/wiki/Spacetime_diagram#Minkowski_diagrams