Relativistic Kinematics

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1 Review

1.1 Lorentz Transformation and Fundamental Results

This review is quite brief. If you are unfamiliar with the topic, please read chapter 11 of [M2].

In special relativity, we deal with *events*, which can be thought of as points in spacetime. Suppose an event has spacetime coordinates (x, t) in an interial frame S. Let S' be an inertial frame moving with velocity $v\hat{\mathbf{x}}$ with respect to S, such that the spacetime origins of the frames coincide (the event $(0, 0)$ in S is the same as the event $(0, 0)$ in S'). Then, the spacetime coordinates (x', t') in the frame S' of the same event are given by

$$t' = \gamma(t - vx/c^2), \quad x' = \gamma(x - vt),$$

where $\gamma := (1 - v^2/c^2)^{-1/2}$. This is known as the *Lorentz transformation.*

Problem 1 (Fundamental Effects)**.** Use the Lorentz transformation to derive the three fundamental results:

- A clock moving with speed v runs a factor of γ slower as viewed by a stationary observer.
- An object of rest length L moving at speed v has length L/γ as viewed by a stationary observer.
- Suppose we have two clocks that are separated by L and are synchronized in their rest frame, but are moving to the right at speed v in our frame. Then, the leftmost clock reads Lv/c^2 higher than the rightmost clock.

Problem 2 (Velocity Addition)**.** Define frames S and S' as above (S' is moving with velocity $v\hat{\mathbf{x}}$ with respect to S). If an object has velocity (u'_x, u'_y) in frame S', then show that the velocity (u_x, u_y) in frame S is given by

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}, \quad u_y = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}.$$

1.2 Introduction to Four-Vectors

A four-vector (X_0, X_1, X_2, X_3) is a set of four physical quantities that transform in the same way as $(c\Delta t, \Delta x, \Delta y, \Delta z)$ under a inertial frame change. The inner product of two four vectors X_i and Y_i is given by

$$X_0Y_0 - X_1Y_1 - X_2Y_2 - X_3Y_3,$$

which can be shown to be invariant under a Lorentz transformation. The most important example of this is that given two events with spacetime displacement $(c\Delta t, \Delta x, \Delta y, \Delta z)$, the quantity

$$(\Delta s)^2 := (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

is the same in all intertial frames.

There are examples of four vectors besides $(c\Delta t, \Delta x, \Delta y, \Delta z)$ such as one for velocity, acceleration, and most importantly energy-momentum, but we won't cover these here (look out for energy-momentum four vector in the relativistic dynamics class).

2 Problems

Problem 3 ([M2] 11.2)**.** Two planets, A and B, are at rest with respect to each other, a distance L apart, with synchronized clocks. A spaceship flies at speed v past planet A toward planet B and synchronizes its clock with A's right when it passes A (they both set their clocks to zero). The spaceship eventually flies past planet B and compares its clock with B's. We know, from working in the planets' frame, that when the spaceship reaches B, B's clock reads L/v. And the spaceship's clock reads $L/\gamma v$, because it runs slow by a factor of γ when viewed in the planets' frame.

How would someone on the spaceship quantitatively explain to you why B's clock reads L/v (which is *more* than its own $L/\gamma v$), considering that the spaceship sees B's clock running *slow*?

Problem 4 ([M2] 11.6)**.** A train and a tunnel both have proper lengths L. The train moves toward the tunnel at speed v. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, the sensor tells the bomb to disarm itself. Does the bomb explode?

Problem 5 (Classical)**.** Define frames S and S' as usual. If a photon is moving at an angle θ with respect to the x'-axis in frame S', what is its angle of motion with respect to the x-axis in the frame S?

Problem 6 ([M2] 11.45)**.** A stick of proper length L moves at speed v along its length. It passes over an infinitesimally thin sheet that has a hole of diameter L cut in it. As the stick passes over the hole, the sheet is raised.

In the lab frame, the stick's length is contracted to L/γ, so it appears to easily make it through the hole. But in the stick frame, the hole is contracted to L/γ, so it appears that the stick does not make it through the hole (or rather, the hole doesn't make it around the stick, since the hole is what is moving in the stick frame). Does the stick end up on the other side of the sheet or not?

3 More Practice

- USAPhO 2016 A3
- IPhO 2006 A2

References

[M2] Morin, David J. *Introduction to Classical Mechanics: With Problems and Solutions.*