Harvard Physics Circle
Lecture 5: Work-Energy

Atınç Çağan Şengül

October 17th, 2020

1 Kinetic Energy & Work

Now we define another useful quantity, *kinetic energy.*

$$E_K = \frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}m\vec{v}\cdot\vec{v} \tag{1}$$

Suppose we move along a path γ from point A to point B. Let us parametrize that path with time, $\vec{r}(t)$

$$\begin{aligned}
E_K(t_B) - E_K(t_A) &= \int_{t_A}^{t_B} dt \left(\frac{dE_K}{dt}\right) && (2)\\
&= \int_{t_A}^{t_B} dt\, m\frac{d\vec{v}}{dt}\cdot\vec{v} && (3)\\
&= \int_{t_A}^{t_B} dt\, \vec{F}\cdot\vec{v} && (4)
\end{aligned}$$

This is by definion a path integral, so we can write,

$$\boxed{E_K(\vec{r}_B) - E_K(\vec{r}_A) = \int_{\substack{A\\ \text{along}\,\gamma}}^{B} d\vec{r}\cdot\vec{F}} \tag{5}$$

This is why kinetic energy is useful. In fact, the right hand side of the integral is also a useful concept so it deserves its own name, *work.*

Problem 1: Calculate and compare the kinetic energy of:

- a running human whose mass is 75 kg and speed is 5 m/s
- a bullet whose mass is 8 g and speed is 715 m/s

Solution 1: The kinetic energy is given by $\frac{1}{2}mv^2$. So we can calculate the kinetic energy of the human and the bullet as,

$$E_{\text{human}} = 0.5(75\,\text{kg})(5\,\text{m/s})^2 = 937.5\,\text{J} \tag{6}$$

$$E_{\text{bullet}} = 0.5(8\times10^{-3}\,\text{kg})(715\,\text{m/s})^2 = 2044.9\,\text{J} \tag{7}$$

The fact that kinetic energy scales with the square of the velocity is what makes projectile weapons, driving over the speed limit, asteroid impacts and other high-speed collisions so deadly.

Problem 2: A 2 kg box rests on the floor. How much work is required to move it at constant speed

- 3m along the floor against a friction force of 4N,
- 3m along a frictionless air table,
- 3m vertically?

Solution 2: We need to apply a force equal to the friction to move the box at a constant speed. So the work we need to do is,

$$W = (3\,\mathrm{m})(4\,\mathrm{N}) = 12\,\mathrm{Nm} \tag{8}$$

We do not need any force to move it in the frictionless case, so the work is 0. For the vertical case we need to match the force of gravity. Therefore the amount of work we need to do is,

$$W = (3\,\mathrm{m})(2\,\mathrm{kg})(9.8\,\mathrm{m/s^2}) = 58.8\,\mathrm{Nm} \tag{9}$$

Problem 3: An incoming ball with mass of $m = 0.4$ kg hits a footballer on his chest with a velocity of $v = 10$ m/s and comes to a stop. During the bounce the footballer's chest compresses by $d = 1$ cm. What is the average force exerted on the footballer?

Solution 3: The chest of the footballer does work to stop the kinetic energy of the incoming ball. So we can write the work-energy equation as,

$$W = F_{\mathrm{avg}} d = \frac{1}{2} m v^2 \quad \Rightarrow \quad F_{\mathrm{avg}} = \frac{mv^2}{2d} = \frac{(0.4\,\mathrm{kg})(10\,\mathrm{m/s})^2}{2(0.01\,\mathrm{m})} = 2000\,\mathrm{N} \tag{10}$$

2 Potential Energy

Something beautiful happens when the path integral that gives work is path independent. It allows us to define, *potential energy.*

$$V(\vec{r}) \equiv - \underset{\text{along any path}}{\int_{\vec{r}_0}^{\vec{r}}} d\vec{r} \cdot \vec{F} \tag{11}$$

Exercise for the ambitious: We can show that
1) The integral of $\vec{F}$ from any point to any other point is path independent.
2) The integral of $\vec{F}$ over any closed path vanishes.
3) $\nabla \times \vec{F} = 0$ everywhere.(hard)
are equivalent statements. These statements are trivially true when our system is one dimensional.

Another way of writing Eq (39) is,

$$V(\vec{r}) - V(\vec{r}_0) = - \underset{\text{along any path}}{\int_{\vec{r}_0}^{\vec{r}}} d\vec{r} \cdot \vec{F} \tag{12}$$

So we can rewrite Eq (38) as,

$$E_K(\vec{r}_B) - E_K(\vec{r}_A) = V(\vec{r}_A) - V(\vec{r}_B) \tag{13}$$

Rearranging terms we can write,

$$\boxed{E_K(\vec{r}_B) + V(\vec{r}_B) = E_K(\vec{r}_A) + V(\vec{r}_A)} \tag{14}$$

We will define this conserved quantity as *total mechanical energy.*

$$E \equiv E_K + V \tag{15}$$

Problem 4: What is the potential energy, due to Earth's gravity, of an object with mass m which is the height h above the Earth's surface? You can assume h is small enough that the gravitational acceleration near Earth is constant and that the gravitational potential is 0 on the Earth's surface.

Solution 4: We need to calculate the work that is needed to lift the object from Earth's surface to height h, which will be equal to its potential energy at that height. We need to apply a force equal to the object's weight and move it vertically by h. So the work is,

$$W = E_p = mgh \tag{16}$$

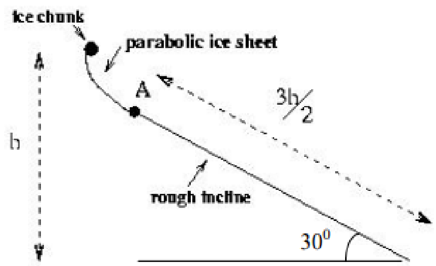
Problem 5: What is the potential energy stored in an elastic spring with the spring constant k, when it is compressed from its relaxed position by the amount x?

Solution 5: Hooke's law gives us that the spring exerts a force resisting the compression that is proportional to the amount of compression. We need to apply a force equal to this while we move by the total amount x. When the spring is compressed by x' we are exerting a force with the amount $F = kx'$. The potential energy is equal to the total work we need to do to compress the spring by the amount x. So we calculate,

$$W = E_p = \int dx' F(x') = \int dx'\, kx' = \frac{1}{2}kx'^2 \tag{17}$$

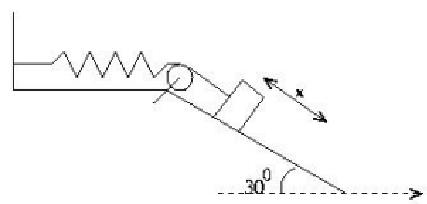
2.1 F=ma 2007

18. A small chunk of ice falls from rest down a frictionless parabolic ice sheet shown in the figure. At the point labeled **A** in the diagram, the ice sheet becomes a steady, rough incline of angle 30^0 with respect to the horizontal and friction coefficient μ_k. This incline is of length $\frac{3}{2}h$ and ends at a cliff. The chunk of ice comes to rest precisely at the end of the incline. What is the coefficient of friction μ_k?



(a) 0.866
(b) 0.770
(c) 0.667
(d) 0.385
(e) 0.333

19. A non-Hookian spring has force $F = -kx^2$ where k is the spring constant and x is the displacement from its unstretched position. For the system shown of a mass m connected to an unstretched spring initially at rest, how far does the spring extend before the system momentarily comes to rest? Assume that all surfaces are frictionless and that the pulley is frictionless as well.



(a) $\left(\frac{3mg}{2k}\right)^{1/2}$

(b) $\left(\frac{mg}{k}\right)^{1/2}$

(c) $\left(\frac{2mg}{k}\right)^{1/2}$

(d) $\left(\frac{\sqrt{3}mg}{k}\right)^{1/3}$

(e) $\left(\frac{3\sqrt{3}mg}{2k}\right)^{1/3}$

3 Power

We can define another useful quantity which is work done per time, which is called *power*. Mathematically, we write it as,

$$P = \frac{\delta W}{dt}. \tag{18}$$

It is measured in J/s which is named Watt and written as W.

Problem 6: An elevator has a motor that can output a maximum of 9800 W. Its cabin weighs 125 kg. How fast can this elevator lift a person weighing 75 kg up 20 m?

Solution 6: The person and the cabin weighs $G = (200\,\text{kg})(9.8\,\text{m/s}) = 1960\,\text{N}$. To move this weight up with a velocity v requires a power of $P = Gv$. Therefore the maximum speed that the elevator can lift is $9800\,\text{W}/1960\,\text{N} = 5\,\text{m/s}$. So it can travel 20 m in 4 seconds.

3.1 F=ma 2008

19. A car has an engine which delivers a constant power. It accelerates from rest at time $t = 0$, and at $t = t_0$ its acceleration is a_0. What is its acceleration at $t = 2t_0$? Ignore energy loss due to friction.

    (a) $\frac{1}{2}a_0$
    (b) $\frac{1}{\sqrt{2}}a_0$
    (c) a_0
    (d) $\sqrt{2}a_0$
    (e) $2a_0$

4 Collisions

4.1 1D Elastic Collision

We will now consider a 1D elastic collision between two particles: 1, and 2. We know momentum and kinetic energy before and after the collision have to be the same.

$$\vec{p}_{1,i} + \vec{p}_{2,i} = \vec{p}_{1,f} + \vec{p}_{2,f} \tag{19}$$

$$E_{1,i} + E_{2,i} = E_{1,f} + E_{2,f} \tag{20}$$

Let the masses and initial velocities of the particles be, $m_1, v_{1,i}, m_2, v_{2,i}$. So we can write these equations as,

$$m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f} \tag{21}$$

$$\frac{1}{2}m_1 v_{1,i}^2 + \frac{1}{2}m_2 v_{2,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2 \tag{22}$$

Using Eq (48), we can write,

$$v_{1,f} - v_{1,i} = -(m_2/m_1)(v_{2,f} - v_{2,i}) \tag{23}$$

When $m_2/m_1 \approx 0$, we see that $v_{1,f} \approx v_{1,i}$. Which makes sense since hitting a table tennis ball would not change a car's velocity. We can also write the energy equation in a simpler form,

$$v_{1,f}^2 - v_{1,i}^2 = -(m_2/m_1)(v_{2,f}^2 - v_{2,i}^2) \tag{24}$$

Plugging Eq (50) into Eq (51) gives,

$$\left(v_{1,i} - (m_2/m_1)(v_{2,f} - v_{2,i})\right)^2 - v_{1,i}^2 = -(m_2/m_1)(v_{2,f}^2 - v_{2,i}^2) \tag{25}$$

let $x = (m_2/m_1)(v_{2,f} - v_{2,i})$, which gives,

$$-2v_{1,i}x + x^2 = -x((m_1/m_2)x + 2v_{2,i}) \tag{26}$$

We can put this in a cleaner quadratic form,

$$x\left[(1 + (m_1/m_2))x + 2(v_{2,i} - v_{1,i})\right] = 0 \tag{27}$$

So $x = 0$ is one of the solutions but it is not interesting since it gives the case when they do not affect each other and continue with their old velocities. So the other solution is,

$$x = -\frac{2(v_{2,i} - v_{1,i})}{1 + (m_1/m_2)} \tag{28}$$

So for $v_{2,f}$ we get,

$$\boxed{v_{2,f} = v_{2,i} - (m_1/m_2)\frac{2(v_{2,i} - v_{1,i})}{1 + (m_1/m_2)}} \tag{29}$$

Similarly for $v_{1,f}$ we get,

$$\boxed{v_{1,f} = v_{1,i} + \frac{2(v_{2,i} - v_{1,i})}{1 + (m_1/m_2)}} \tag{30}$$

These expressions look rather crowded. However, they actually have a very simple interpretation. Let us solve the same problem, but this time we will work in the center of mass frame. The center of mass velocity of the system can be written as,

$$\beta = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{(m_1/m_2)v_1 + v_2}{1 + (m_1/m_2)} \tag{31}$$

Let us denote the velocities of the particles in the center of mass frame as $u = v - \beta$. So we have,

$$u_1 = v_1 - \beta = \frac{(1 + (m_1/m_2))v_1 - (m_1/m_2)v_1 - v_2}{1 + (m_1/m_2)} = \frac{v_1 - v_2}{1 + (m_1/m_2)} \tag{32}$$

$$u_2 = v_2 - \beta = (m_1/m_2)\frac{v_2 - v_1}{1 + (m_1/m_2)} = -(m_1/m_2)u_1 \tag{33}$$

Since there are no external forces we know that $\beta_i = \beta_f$. So momentum conservation is trivially satisfied.

$$u_{1,f} - u_{1,i} = -(m_2/m_1)(u_{2,f} - u_{2,i}) \quad \Rightarrow \quad u_{1,f} - u_{1,i} = (m_2/m_1)(m_1/m_2)(u_{1,f} - u_{1,i}) \tag{34}$$

The energy conservation becomes,

$$u_{1,f}^2 - u_{1,i}^2 = -(m_2/m_1)(u_{2,f}^2 - u_{2,i}^2) \quad \Rightarrow \quad u_{1,f}^2 - u_{1,i}^2 = -(m_2/m_1)(m_1/m_2)^2(u_{1,f}^2 - u_{1,i}^2) \tag{35}$$

$$\Rightarrow \quad \boxed{u_{1,f}^2 = u_{1,i}^2} \quad \boxed{u_{2,f}^2 = u_{2,i}^2} \tag{36}$$

So in the center of mass frame, the particle velocities either do not change at all (no collision), or they stay the same value but flip sign. Flipping sign in an arbitrary frame means subtracting twice the velocity relative to the center of mass. We can see this in Eq (56) and (57).

$$\boxed{v_{2,f} = v_{2,i} - 2u_{2,i}} \quad \boxed{v_{1,f} = v_{1,i} - 2u_{1,i}} \tag{37}$$

Exercise: Study the $m_2/m_1 \to \infty$ and $m_2/m_1 \to 0$ limits.

4.2 2D Collision

We now consider the collision of two particles like before, but now in a 2 dimensional space.
Exercise: Prove that one can always choose a 2D plane such that a collision of 2 particles in 3D space is restricted to that 2D plane.
Without loss of generality we can choose $m_2 \geq m_1,\ \vec{v}_{2,i} = 0,\ \vec{v}_{1,i} = v\hat{x}$. (**Exercise:** Why?)

$$m_1 v\hat{x} = m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f} \tag{38}$$

$$m_1 v^2 = m_1|\vec{v}_{1,f}|^2 + m_2|\vec{v}_{2,f}|^2 \tag{39}$$

Let us now denote $\vec{v}_{1,f} = (v_{1x}, v_{1y}), \vec{v}_{2,f} = (v_{2x}, v_{2y})$. So we get 3 scalar equations.

$$m_1 v = m_1 v_{1x} + m_2 v_{2x} \tag{40}$$

$$0 = m_1 v_{1y} + m_2 v_{2y} \tag{41}$$

$$m_1 v^2 = m_1(v_{1x}^2 + v_{1y}^2) + m_2(v_{2x}^2 + v_{2y}^2) \tag{42}$$

We notice that there a 3 equations and 4 variables to solve for. This means there is a continuum of solutions characterized by a free parameter. In other words we need an additional piece of information to solve the problem fully. Suppose we also know the angle that $\vec{v}_{1,f}$ makes with the x-axis, θ. So we can write,

$$v_{1y} = v_1 \sin\theta, \quad v_{1x} = v_1 \cos\theta \tag{43}$$

Using this, and Eq (66), we have,

$$v = v_1 \cos\theta + (m_2/m_1)v_{2x} \tag{44}$$

$$v^2 = v_1^2 + (m_2/m_1)v_{2x}^2 + (m_1/m_2)v_1^2 \sin^2\theta \tag{45}$$

Using the first equation to substitute v_{2x}, we get,

$$v^2 = v_1^2 + (m_1/m_2)[v^2 + v_1^2\cos^2\theta - 2vv_1\cos\theta] + (m_1/m_2)v_1^2\sin^2\theta \tag{46}$$

Which can be simplified into,

$$\left(1 + \frac{m_1}{m_2}\right) v_1^2 + \left(-2\frac{m_1}{m_2} v\cos\theta\right) v_1 - \left(1 - \frac{m_1}{m_2}\right) v^2 = 0 \tag{47}$$

Let us denote $\gamma = m_1/m_2$. The solution to this is,

$$v_1 = v\frac{\gamma\cos\theta \pm \sqrt{1 - \gamma^2\sin^2\theta}}{(1+\gamma)} \tag{48}$$

We notice that,

$$\gamma^2\cos^2\theta \leq 1 - \gamma^2\sin^2\theta, \quad \text{because} \quad \gamma^2 \leq 1 \tag{49}$$

The negative solution therefore can be absorbed into choosing $\theta \to \theta + \pi$. So we have,

$$\boxed{v_1 = v\frac{\gamma\cos\theta + \sqrt{1 - \gamma^2\sin^2\theta}}{(1+\gamma)}} \tag{50}$$

Exercise: Study what happens when $\theta \to 0, \pi \quad \& \quad \gamma \to 1, 0$

4.3 F=ma 2009

1. A 0.3 kg apple falls from rest through a height of 40 cm onto a flat surface. Upon impact, the apple comes to rest in 0.1 s, and 4 cm^2 of the apple comes into contact with the surface during the impact. What is the average pressure exerted on the apple during the impact? Ignore air resistance.

    (A) 67,000 Pa

    (B) 21,000 Pa

    (C) 6,700 Pa

    (D) 210 Pa

    (E) 67 Pa

    The following information is used for questions 2 and 3.

    Three blocks of identical mass are placed on a frictionless table as shown. The center block is at rest, whereas the other two blocks are moving directly towards it at identical speeds v. The center block is initially closer to the left block than the right one. All motion takes place along a single horizontal line.

2. Suppose that all collisions are instantaneous and perfectly elastic. After a long time, which of the following is true?

    (A) The center block is moving to the left.

    (B) The center block is moving to the right.

    (C) The center block is at rest somewhere to the left of its initial position.

    (D) The center block is at rest at its initial position.

    (E) The center block is at rest somewhere to the right of its initial position.

3. Suppose, instead, that all collisions are instantaneous and perfectly inelastic. After a long time, which of the following is true?

    (A) The center block is moving to the left.

    (B) The center block is moving to the right.

    (C) The center block is at rest somewhere to the left of its initial position.

    (D) The center block is at rest at its initial position.

    (E) The center block is at rest somewhere to the right of its initial position.

5 Homework and Additional Exercises

You can solve more of the past F=ma problems that are related to work-energy. You can find the list of topics of each problem in the blog of Kevin S Huang that is given in the references. For more challanging problems you can look at sections 5.9 and 5.10 of David Morin's book which is also given in the references.

6 References

- https://kevinshuang.com/2019/12/28/2019-fma-exam-analysis/
- https://www.aapt.org/physicsteam/2018/exams.cfm
- David Morin, *Introduction to Classical Mechanics With Problems and Solutions*